Module 2-4: Introduction to Adaptive Control

Linear Control Systems (2020)

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Motivation

"All models are wrong, but some are useful."– George Box, statistician
Uncertainty and error in modeling are inevitable in practice. We may not fully understand the dynamics of the system or the parameters of the system are changing over time. Examples:

- The vehicle dynamics are difficult to model 100% accurately due to complexity.
- As an airplane flies, its mass is decreasing due to fuel consumption.

How to maintain consistent performance of a system in the presence of model uncertainty or unknown parameter variation?

Use adaptive control!

A Brief History

- With the emergence of servosystems and flight controllers in 1950s, adaptive control became an important discipline in control and dynamics. The initial motivation can be traced to the design of autopilots for high-performance aircraft.
- The last several decades have witnessed the development on both the theoretical side and the application side. Early advances in system identification and dynamic programming have constructed the foundations for adaptive and learning control.
- Industrial applications include chemical reactor control, engine control, ship/aircraft autopilot, power plant control, ...

Classification of Adaptive Control Methods

1. Model-based Adaptive Control
    - Fully Model-based, e.g., Model Reference Adaptive Control (MRAC).
      Very mature in terms of theoretical guarantees. However restricted to some know types of models due to the model-based formulations.
    - Learning-based, e.g., model-based reinforcement learning controller.
      Partially model-based. Unmodeled part is handled by some data-driven optimization and learning algorithms, therefore gain flexibility.
2. Data-Driven Adaptive Control
    - e.g., Model-free reinforcement learning controller.
      Work without any prior knowledge about the model. Great flexibility. However requiring extensive measurements and collecting data. In addition, it lacks stability and performance guarantees.

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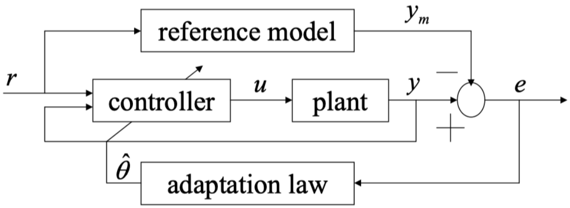
Model-Reference Adaptive Control (MRAC)

- **MRAC** is one of the fully model-based adaptive controllers. Model-based adaptive controller can be viewed as a dynamic system with online parameter estimation.
- Indirect Adaptive Control: compute controller parameters by estimating plant parameters, therefore relies on convergence of the estimated parameters to their true values.
- Direct Adaptive Control: estimate the controller parameters directly without estimating plant parameters.
- We will focus on Direct MRAC design and analysis.



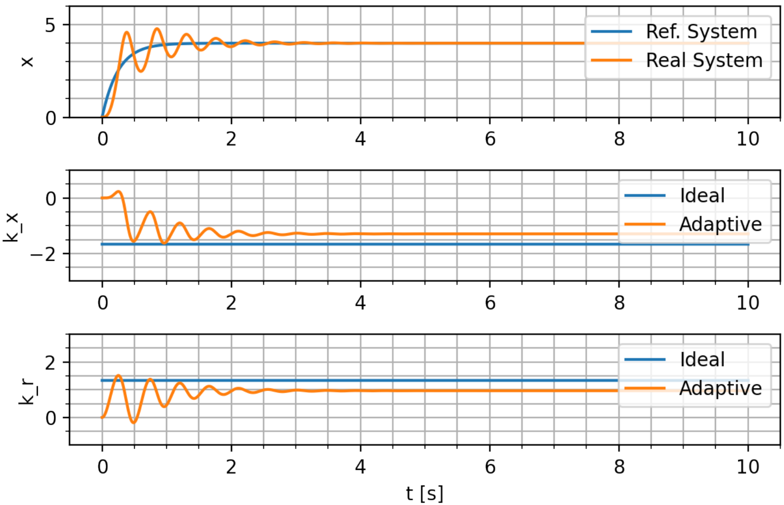
Courtesy: Lavretsky, E.

Example: MRAC for a 1st Order Linear System

- Unknown dynamics: $\dot{x} = ax + bu, \quad x(0) = 0$. The true model parameter $a = 1, b = 3$ are unknown to the adaptive controller
- Reference (desired) model: $\dot{x}_m = -4x_m + 4r(t), \quad x_m(0) = 0$
- The adaptive controller: $u = \hat{k}_x x + \hat{k}_r r$
- The adaptive law (based on Lyapunov theory)
  - $e = x - x_m$
  - $\dot{\hat{k}}_x = -2xe, \quad \hat{k}_x(0) = 0$
  - $\dot{\hat{k}}_r = -2re, \quad \hat{k}_r(0) = 0$
- Two different reference signals:
  - $r(t) = 4$
  - $r(t) = 4\sin(3t)$

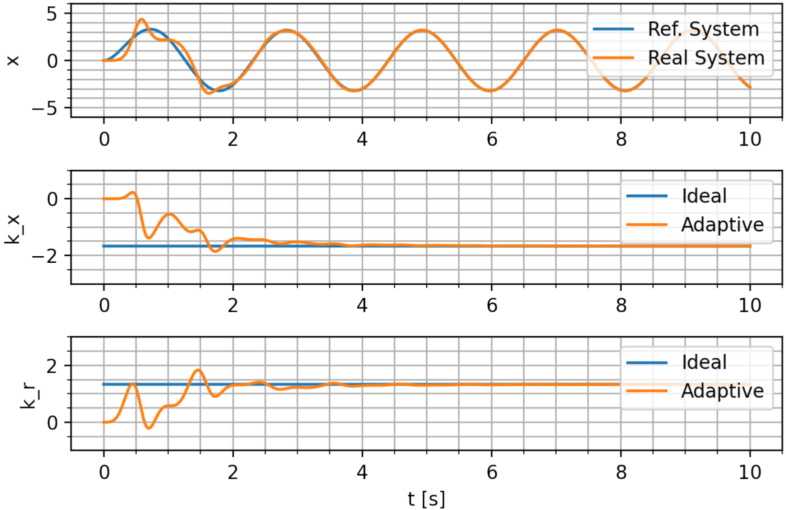
Example: MRAC for a 1st Order Linear System (continued)

No Persistency of Excitation: $r(t) = 4$



Example: MRAC for a 1st Order Linear System (continued)

Persistency of Excitation: $r(t) = 4\sin(3t)$



MRAC for 1st Order Systems

- 1st Order Systems: $\dot{x} = ax + b(u + f(x))$, where a,b are constant unknown parameters. We assumed that sign of b is known. The unknown nonlinear function $f(x)$ is linearly parameterized by N unknown constant parameters θ_i and known basis functions φ_i, i.e.,

$$f(x) = \sum_{i=1}^{N} \theta_i \varphi_i(x) = \theta^T \Phi(x)$$

  where $\theta = \begin{pmatrix} \theta_1 & \dots & \theta_N \end{pmatrix}^T$ and $\Phi(x) = (\varphi_1(x) \quad \dots \quad \varphi_N(x))^T$
- A reference model is described by the 1st order differential equation:

$$\dot{x}_m = a_m x_m + b_m r(t) \tag{1}$$

  where $a_m < 0$ and b_m are the known desired constants and $r(t)$ is the reference input.
- Control goal: Design a controller $u(t)$ such that all signals in the system remain bounded, and the tracking error $e(t) = x(t) - x_m(t) \to 0$ as $t \to \infty$.

MRAC for 1st Order Systems (cont'd)

- An ideal control solution formed using feedback and feedforward architecture. If we knew the unknown parameters, we could directly design the controller as

$$u_{\text{ideal}} = k_x x + k_r r(t) - \theta^T \Phi(x) \tag{2}$$

- Substitute (2) into the system equation, the closed-loop dynamics becomes:

$$\dot{x} = (a + bk_x)\, x + bk_r r(t) \tag{3}$$

- Compare (3) with the reference model (1), the ideal gains k_x and k_r must satisfy the following matching conditions:

$$\begin{aligned} a + bk_x &= a_m \\ bk_r &= b_m \end{aligned} \tag{4}$$

- Obviously ideal gains k_x and k_r always exist by solving (4). However, in reality, a, b, θ are unknown.

MRAC for 1st Order Systems (cont'd)

- We form a control solution similar to (2):

$$u = \hat{k}_x x + \hat{k}_r r(t) - \hat{\theta}^T \Phi(x) \tag{5}$$

We need to find the feedback gain $\hat{k}_x$, the feedforward gain $\hat{k}_r$, and the estimated vector of parameters $\hat{\theta}$ to achieve desired tracking of the reference model.

- Substitute (5) into the system equation:

$$\dot{x} = \left(a + b\hat{k}_x\right) x + b\left(\hat{k}_r r(t) - (\hat{\theta} - \theta)^T \Phi(x)\right) \tag{6}$$

- Substitute (4) into (6):

$$\dot{x} = a_m x + \underbrace{bk_r}_{b_m} r(t) + b \underbrace{\left(\hat{k}_x - k_x\right)}_{\Delta k_x} x + b \underbrace{\left(\hat{k}_r - k_r\right)}_{\Delta k_r} r(t) - b \underbrace{\left(\hat{\theta} - \theta\right)^T}_{\Delta \theta^T} \Phi(x) \tag{7}$$

MRAC for 1st Order Systems (cont'd)

- The closed-loop dynamics of the tracking error $e(t) = x(t) - x_m(t)$ can be obtained by subtracting (1) from (7):

$$\dot{e} = \dot{x} - \dot{x}_m = a_m e + b\left(\Delta k_x x + \Delta k_r r - \Delta\theta^T \Phi(x)\right)$$

Because $a_m < 0$, $x_m(t)$ is a bounded function of $r(t)$. Denote $x_m(t) = x_m(r)$

$$\dot{e} = a_m e + b\left(\Delta k_x (e + x_m(r)) + \Delta k_r r - \Delta\theta^T \Phi(x)\right) \tag{8}$$

- Consider the Lyapunov function candidate:

$$V\left(e, \Delta k_x, \Delta k_r, \Delta\theta\right) = e^2 + |b|\left(\gamma_x^{-1}\Delta k_x^2 + \gamma_r^{-1}\Delta k_r^2 + \Delta\theta^T \Gamma_\theta^{-1} \Delta\theta\right)$$

where $\gamma_x > 0, \gamma_r > 0$, and $\Gamma_\theta = \Gamma_\theta^T > 0$ are rates of adaptation.

MRAC for 1st Order Systems (cont'd)

Take time derivative of V, along the trajectories of (8):

$$\dot{V}(e, \Delta k_x, \Delta k_r, \Delta\theta) = 2e\dot{e} + 2|b|\left(\gamma_x^{-1}\Delta k_x \dot{\hat{k}}_x + \gamma_r^{-1}\Delta k_r \dot{\hat{k}}_r + \Delta\theta^T \Gamma_\theta^{-1}\dot{\hat{\theta}}\right)$$

$$= 2e\left(a_m e + b\left(\Delta k_x x + \Delta k_r r - \Delta\theta^T \Phi(x)\right)\right) + 2|b|\left(\gamma_x^{-1}\Delta k_x \dot{\hat{k}}_x + \gamma_r^{-1}\Delta k_r \dot{\hat{k}}_r + \Delta\theta^T \Gamma_\theta^{-1}\dot{\hat{\theta}}\right)$$

$$= 2a_m e^2 + 2|b|\left(\Delta k_x\left(xe\,\mathrm{sign}(b) + \gamma_x^{-1}\dot{\hat{k}}_x\right)\right)$$

$$+ 2|b|\left(\Delta k_r\left(re\,\mathrm{sign}(b) + \gamma_r^{-1}\dot{\hat{k}}_r\right)\right) + 2|b|\Delta\theta^T\left(-\Phi(x)e\,\mathrm{sign}(b) + \Gamma_\theta^{-1}\dot{\hat{\theta}}\right)$$

If we choose the adaptive laws:

$$\dot{\hat{k}}_x = -\gamma_x x e\,\mathrm{sign}(b)$$

$$\dot{\hat{k}}_r = -\gamma_r r e\,\mathrm{sign}(b)$$

$$\dot{\hat{\theta}} = \Gamma_\theta \Phi(x) e\,\mathrm{sign}(b)$$

The time derivative of V becomes $\dot{V}(e, \Delta k_x, \Delta k_r, \Delta\theta) = 2a_m e(t)^2 \le 0$

Summary of What We Have So Far

We want

$$\dot{x} = ax + b(u + \theta^T \Phi(x))$$

behaves like

$$\dot{x}_m = a_m x_m + b_m r(t)$$

but do not know a, b, θ except the sign of b. Propose control law

$$u = \hat{k}_x x + \hat{k}_r r(t) - \hat{\theta}^T \Phi(x)$$

$$\dot{\hat{k}}_x = -\gamma_x x e \operatorname{sign}(b)$$

$$\dot{\hat{k}}_r = -\gamma_r r e \operatorname{sign}(b)$$

$$\dot{\hat{\theta}} = \Gamma_\theta \Phi(x) e \operatorname{sign}(b)$$

Tracking error dynamics $e = x - x_m$:

$$\dot{e} = a_m e + b\left(\Delta k_x (e + x_m(r)) + \Delta k_r r - \Delta\theta^T \Phi(x)\right)$$

Lyapunov function
$V\left(e, \Delta k_x, \Delta k_r, \Delta\theta\right) =$
$e^2 + |b|\left(\gamma_x^{-1} \Delta k_x^2 + \gamma_r^{-1} \Delta k_r^2 + \Delta\theta^T \Gamma_\theta^{-1} \Delta\theta\right) > 0$

$$\dot{V}\left(e, \Delta k_x, \Delta k_r, \Delta\theta\right) = 2 a_m e(t)^2 \leq 0$$

Now, we show that $e, \Delta k_x, \Delta k_r, \Delta\theta$ are bounded, but we do not know whether they will converge to $\mathbf{0}$, because the system has an input $r(t)$ (non-autonomous).

Barbalat's lemma

Lyapunov direct method

The origin of $\dot{x} = f(x)$ is stable if $\exists V(x,t)$

1. $V(x) = 0$ when $x = 0$
2. $V(x) > 0$ when $x \neq 0$
3. $\dot{V}(x) \leq 0$

Note: Lyapunov only applies to autonomous systems (no input).

Barbalat's lemma

Given $\dot{x} = f(x,u)$ if $\exists V(x,t)$

1. $V(x,t) = 0$ when $x = 0$
2. $V(x,t) > 0, \forall x \neq 0$
3. $\dot{V}(x,t) \leq 0, \forall x$
4. $\lim_{t\to\infty} \ddot{V}(x,t)$ bounded

Then $\lim_{t\to\infty} \dot{V}(x,t) = 0$

Note 1: Basically, Barbalat's lemma shows that if both $V(x)$ and $\ddot{V}(x,t)$ are bounded, then $\lim_{t\to\infty} \dot{V}(x,t) = 0$.

Note 2: $V(x,t), \dot{V}(x,t), \ddot{V}(x,t)$ have t because it may have $r(t)$ in the expression, which makes the system time-varying.

Convergence of MRAC

- $V \geq 0$ and $\dot{V} \leq 0 \Rightarrow e$, Δk_x, Δk_r, $\Delta\theta$ are bounded.
- $r(t)$ is bounded $\Rightarrow \dot{x}_m(t)$ and $x_m(t)$ are bounded.
- $x(t) = x_m(t) + e(t) \Rightarrow x(t)$ is bounded.
- Consequently, $u(t)$ is bounded and $\dot{x}(t)$ is bounded as well.
- Therefore $\dot{e}(t)$ is bounded. $\ddot{V}(e, \Delta k_x, \Delta k_r, \Delta\theta) = 4a_m e(t)\dot{e}(t)$ is bounded. By Barbalat's Lemma,

$$\lim_{t\to\infty} \dot{V}(x,t) = 2a_m e(t)^2 = 0$$

We can conclude

$$\lim_{t\to\infty} e(t) = 0$$

Note: we cannot prove $\Delta k_x, \Delta k_r, \Delta\theta \to 0$. Actually, whether they could coverage or not depends on $r(t)$!

Parameter Convergence

- The estimated parameters do not always converge to their true (or ideal) values. It depends on the reference signal $r(t)$.
- A sufficient condition for parameter convergence is that reference signal $r(t)$ satisfies **Persistency of Excitation** (PE). However, PE is difficult to verify.
- Direct MRAC provides good tracking even if the parameters do not converge to their true (or ideal) values.

Example: MRAC for a 1st Order Linear System

- Unknown dynamics: $\dot{x} = ax + bu, \quad x(0) = 0$. The true model parameter $a = 1, b = 3$ are unknown to the adaptive controller
- Reference (desired) model: $\dot{x}_m = -4x_m + 4r(t), \quad x_m(0) = 0$
- The adaptive controller: $u = \hat{k}_x x + \hat{k}_r r$
- The adaptive law (based on Lyapunov theory)
  - $\dot{\hat{k}}_x = -2xe, \quad \hat{k}_x(0) = 0$
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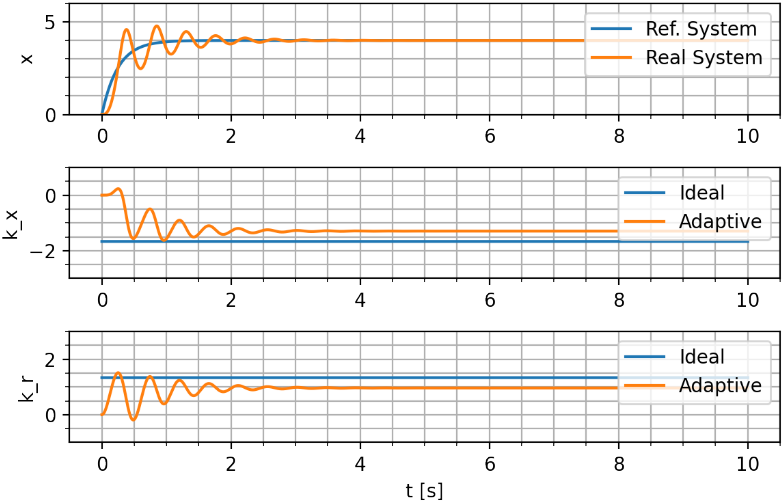
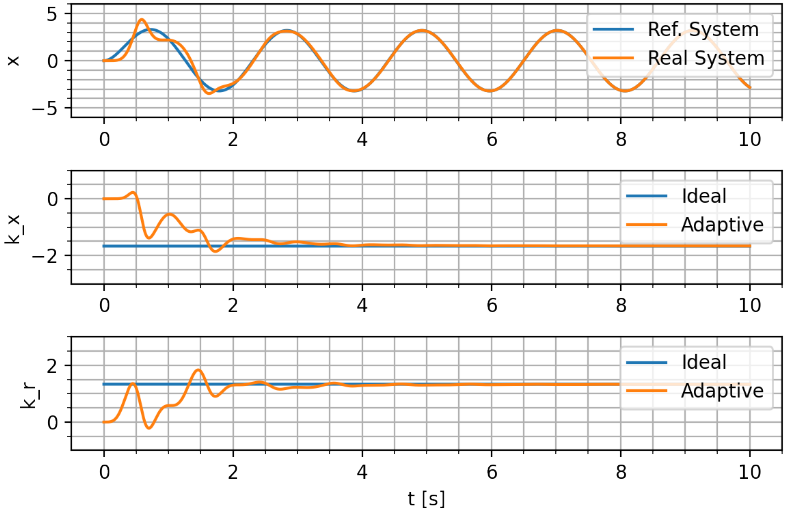
- Two different reference signals:
  - No Persistency of Excitation: $r(t) = 4$
  - Persistency of Excitation: $r(t) = 4\sin(3t)$

Example: MRAC for a 1st Order Linear System (continued)

No Persistency of Excitation: $r(t) = 4$



Persistency of Excitation: $r(t) = 4\sin(3t)$



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Direct MRAC Design for Nonlinear MIMO System

- Consider a special MIMO nonlinear system: $\dot{x} = Ax + B\Lambda(u + f(x))$, where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^M$. $B \in \mathbb{R}^{n\times M}$ is known. $A \in \mathbb{R}^{n\times n}$ and $\Lambda \in \mathbb{R}^{M\times M}$ are unknown matrices. It is assumed that Λ is diagonal with positive elements λ_i, and the pair $(A, B\Lambda)$ is controllable.
- $f(x) : \mathbb{R}^n \to \mathbb{R}^M$ can be written as a linear combination of N known basis functions, with unknown constant matrix $\Theta \in \mathbb{R}^{N\times M}$. $\Phi(x) \in \mathbb{R}^N$ is the basis function vector.

$$f(x) = \Theta^T \Phi(x)$$

- A reference model described by $\dot{x}_m = A_m x_m + B_m r(t)$. The control goal again is to let the state x track x_m
- The ideal control law is $u_{\text{ideal}} = K_x^T x + K_r^T r - \Theta^T \Phi(x)$, as if the unknown matrices were known. The closed loop system is $\dot{x} = \left(A + B\Lambda K_x^T\right) x + B\Lambda K_r^T r$. The matching conditions are $A + B\Lambda K_x^T = A_m$ and $B\Lambda K_r^T = B_m$
Note: K_x and K_r may not exist to satisfy the matching conditions. In practice, A_m and B_m are chosen such that there exists a solution for K_x and K_r.

Direct MRAC Design for Nonlinear MIMO System (cont'd)

- Consider the control law: $u = \hat{K}_x^T x + \hat{K}_r^T r - \hat{\Theta}^T \Phi(x)$
- Similar to the linear case, we have error dynamics:

$$\dot{e} = A_m e + B\Lambda \left[\Delta K_x^T x + \Delta K_r^T r - \Delta\Theta^T \Phi(x)\right]$$

where $\Delta K_x = \hat{K}_x - K_x, \Delta K_r = \hat{K}_r - K_r$, and $\Delta\Theta = \hat{\Theta} - \Theta$.

- Consider the Lynapunov function candidate:

$$V\left(e, \Delta K_x, \Delta K_r, \Delta\Theta\right) = e^T P e + \mathrm{tr}\left(\left[\Delta K_x^T \Gamma_x^{-1} \Delta K_x + \Delta K_r^T \Gamma_r^{-1} \Delta K_r + \Delta\Theta^T \Gamma_\Theta^{-1} \Delta\Theta\right] \Lambda\right)$$

where $P = P^T > 0$ satisfies the algebraic Lyapunov equation $PA_m + A_m^T P = -Q$ for some chosen $Q = Q^T > 0$. $\Gamma_x = \Gamma_x^T > 0$, $\Gamma_r = \Gamma_r^T > 0$, $\Gamma_\Theta = \Gamma_\Theta^T > 0$ are the rates of adaptation.

Direct MRAC Design for Nonlinear MIMO System (cont'd)

$$\begin{aligned}\dot{V} &= \dot{e}^T P e + e^T P \dot{e} + 2\,\mathrm{tr}\left(\left[\Delta K_x^T \Gamma_x^{-1} \dot{\hat{K}}_x + \Delta K_r^T \Gamma_r^{-1} \dot{\hat{K}}_r + \Delta \Theta^T \Gamma_\Theta^{-1} \dot{\hat{\Theta}}\right] \Lambda\right) \\ &= \left(A_m e + B\Lambda\left(\Delta K_x^T x + \Delta K_r^T r - \Delta\Theta^T \Phi(x)\right)\right)^T P e \\ &\quad + e^T P \left(A_m e + B\Lambda\left(\Delta K_x^T x + \Delta K_r^T r - \Delta\Theta^T \Phi(x)\right)\right) \\ &\quad + 2\,\mathrm{tr}\left(\left[\Delta K_x^T \Gamma_x^{-1} \dot{\hat{K}}_x + \Delta K_r^T \Gamma_r^{-1} \dot{\hat{K}}_r + \Delta \Theta^T \Gamma_\Theta^{-1} \dot{\hat{\Theta}}\right] \Lambda\right) \\ &= e^T \left(A_m^T P + P A_m\right) e + 2 e^T P B \Lambda \left(\Delta K_x^T x + \Delta K_r^T r - \Delta\Theta^T \Phi(x)\right) \\ &\quad + 2\,\mathrm{tr}\left(\left[\Delta K_x^T \Gamma_x^{-1} \dot{\hat{K}}_x + \Delta K_r^T \Gamma_r^{-1} \dot{\hat{K}}_r + \Delta \Theta^T \Gamma_\Theta^{-1} \dot{\hat{\Theta}}\right] \Lambda\right)\end{aligned}$$

Since $P A_m + A_m^T P = -Q$,

$$\dot{V} = -e^T Q e + \left[2 e^T P B \Lambda \Delta K_x^T x + 2\,\mathrm{tr}\left(\Delta K_x^T \Gamma_x^{-1} \dot{\hat{K}}_x \Lambda\right)\right]$$

$$+ \left[2 e^T P B \Lambda \Delta K_r^T r + 2\,\mathrm{tr}\left(\Delta K_r^T \Gamma_r^{-1} \dot{\hat{K}}_r \Lambda\right)\right] + \left[-2 e^T P B \Lambda \Delta \Theta^T \Phi(x) + 2\,\mathrm{tr}\left(\Delta \Theta^T \Gamma_\Theta^{-1} \dot{\hat{\Theta}} \Lambda\right)\right]$$

Direct MRAC Design for Nonlinear MIMO System (cont'd)

Since $\mathrm{tr}\left(\mathbf{b}\mathbf{a}^T\right) = \mathbf{a}^T\mathbf{b}$,

$$
\begin{aligned}
\underbrace{e^T PB\Lambda}_{a^T}\underbrace{\Delta K_x^T x}_{b} &= \mathrm{tr}(\underbrace{\Delta K_x^T x}_{b}\underbrace{e^T PB\Lambda}_{a^T}) \\
\underbrace{e^T PB\Lambda}_{a^T}\underbrace{\Delta K_r^T r}_{b} &= \mathrm{tr}(\underbrace{\Delta K_r^T r}_{b}\underbrace{e^T PB\Lambda}_{a^T}) \\
\underbrace{e^T PB\Lambda}_{a^T}\underbrace{\Delta\Theta^T\Phi(x)}_{b} &= \mathrm{tr}(\underbrace{\Delta\Theta^T\Phi(x)}_{b}\underbrace{e^T PB\Lambda}_{a^T})
\end{aligned}
\tag{9}
$$

Substitute (9) into $\dot{V}$

$$
\begin{aligned}
\dot{V} = &-e^T Q e + 2\,\mathrm{tr}\left(\Delta K_x^T\left[\Gamma_x^{-1}\dot{\hat{K}}_x + x e^T PB\right]\Lambda\right) \\
&+ 2\,\mathrm{tr}\left(\Delta K_r^T\left[\Gamma_r^{-1}\dot{\hat{K}}_r + r e^T PB\right]\Lambda\right) + 2\,\mathrm{tr}\left(\Delta\Theta^T\left[\Gamma_\Theta^{-1}\dot{\hat{\Theta}} - \Phi(x) e^T PB\right]\Lambda\right)
\end{aligned}
$$

Direct MRAC Design for Nonlinear MIMO System (cont'd)

Therefore, the adaptive laws are chosen to be

$$\dot{\hat{K}}_x = -\Gamma_x x e^T P B$$

$$\dot{\hat{K}}_r = -\Gamma_r r(t) e^T P B$$

$$\dot{\hat{\Theta}} = \Gamma_\Theta \Phi(x) e^T P B$$

The time-derivative of V becomes negative semi-definite:

$$\dot{V} = -e^T Q e \leq 0$$

The rest of analysis is similar to the SISO case with Barbalat's lemma.

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