Module 1-4: Realization

Linear Control Systems (2020)

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Mathemathical Expressions of Linear Time-Invariant Control Systems

- Ordinary differential equations

$$\begin{aligned} \dot{\mathbf{x}} &= f(\mathbf{x}, \mathbf{u}) \\ \mathbf{y} &= g(\mathbf{x}, \mathbf{u}) \end{aligned}$$

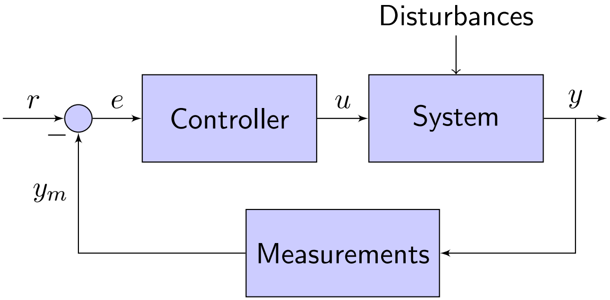
- State space equations

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \end{aligned}$$

- Transfer functions

$$\frac{Y(s)}{U(s)} = H(s)$$

Feedback Control System



Review: The Laplace Transform

Laplace, 1749-1827

The Laplace transform of a signal (function) f is the function $F = \mathscr{L}(f)$ defined by

$$F(s) = \int_0^\infty f(t)e^{-st}dt$$

for those $s \in \mathbb{C}$ for which the integral converges.

- F is a complex-valued function of complex numbers
- s is called the (complex) *frequency variable*, with units $\sec^{-1}$; t is called the *time variable* (in $\sec$); st is unitless

Common notation convention: lower case letter denotes signal. capital letter denotes its Laplace transform e.g. $U(s)$ denotes $\mathscr{L}(u), V_{\text{in}}(s)$ denotes $\mathscr{L}(v_{\text{in}})$, etc.

Example: Exponential Function

Let's find Laplace transform of $f(t) = e^{at}$:

$$F(s) = \int_0^\infty e^{at}e^{-st}dt = \int_0^\infty e^{(a-s)t}dt = \left[\frac{1}{a-s}e^{(a-s)t}\right]_0^\infty = \frac{1}{s-a}$$

Example: Impulse Function

Impulse function/(Dirac) delta function

$$\delta(x) = \begin{cases} +\infty, & x = 0 \\ 0, & x \neq 0 \end{cases}, \qquad \int_{-\infty}^{\infty} \delta(x)\, dx = 1$$

The Laplace Transformation:

$$F(s) = \int_{0-}^{\infty} \delta(t) e^{-st} dt = e^{-st}|_{t=0} = 1$$

Linearity

The Laplace transform is *linear*: If f and g are any signals, and a is any scalar, we have

$$\mathscr{L}(af) = aF, \ \mathscr{L}(f+g) = F + G$$

$i.e.$ homogeneity & superposition hold

Example:

$$\begin{aligned}\mathscr{L}(3\delta(t) - 2e^t) &= 3\mathscr{L}(\delta(t)) - 2\mathscr{L}(e^t) \\ &= 3 - \frac{2}{s-1} \\ &= \frac{3s-5}{s-1}\end{aligned}$$

Review: Derivative and Integral

Let y be the running derivative of a signal u

$$y(t) = \frac{du(t)}{dt}$$

then

$$Y(s) = sU(s)$$

Assume the initial condition $u^{(k)}(0) = 0$.

Let y be the running integral of a signal u

$$y(t) = \int_0^t u(\tau) d\tau$$

then

$$Y(s) = \frac{1}{s} U(s)$$

Convolution Systems

Convolution system wlth input $u(u(t) = 0,\ t < 0)$ and output y:

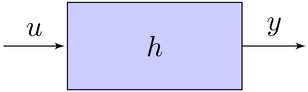
$$y(t) = \int_0^t h(\tau)u(t-\tau)d\tau = \int_0^t h(t-\tau)u(\tau)d\tau$$

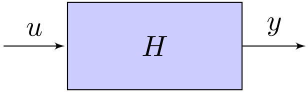
abbreviated: $y = h * u$

In the frequency domain: $\mathrm{Y}(s) = H(s)U(s)$

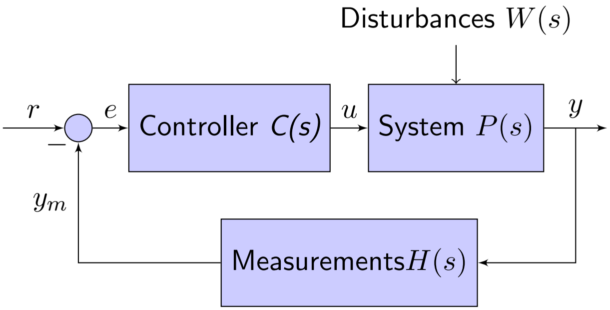
- H is the *transfer function* (TF) of the system
- h is the *impulse response* of the system

block diagram notation(s):





Feedback Control System



$$Y(s) = P(s)C(s)E(s) = P(s)C(s)(R(s) - Y_m(s)) = P(s)C(s)(R(s) - H(s)Y(s))$$

$$(1 + P(s)C(s)H(s))Y(s) = P(s)C(s)R(s)$$

$$\frac{Y(s)}{R(s)} = \frac{P(s)C(s)}{1 + P(s)C(s)H(s)}$$

Review: Inverse Laplace Transform

In principle we can recover f from F via

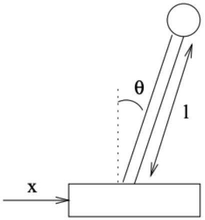
$$f(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} F(s) e^{st} ds$$

where $\Re s \geq \sigma$. Note: If $u(t) = \delta(t), U(s) = \mathscr{L}(\delta(t)) = 1$

$$y(t) = \mathscr{L}^{-1}(H(s) \cdot U(s)) = \mathscr{L}^{-1}(H(s) \cdot 1) = \mathscr{L}^{-1}(H(s))$$
$$H(s) = \mathscr{L}(y(t))$$

Therefore, to learn to TF of a system, we just need to measure the system response of a impulse input and take the Laplace transformation of the output. We call $y(t)$ the impulse response of a system, usually denoted as $h(t)$.

Example: Inverted Pendulum



$$\ddot{\theta} = \ddot{\theta}_g + \ddot{\theta}_x = (g/l)\sin\theta - (\ddot{x}/l)\cos\theta$$

$$l\ddot{\theta} - g\theta = -\ddot{x}$$
$$G(s) = \frac{\Theta(s)}{X(s)} = \frac{-s^2}{ls^2 - g} = \frac{-a^2 s^2/g}{(s+a)(s-a)}$$

where $a = \sqrt{g/l}$.

Example: Inverted Pendulum (continued)

If we assume a impulse input, by inverse Laplace transform, we have

$$
\begin{aligned}
\theta(t) &= \mathscr{L}^{-1}(H(s)\cdot 1) \\
&= -\frac{a^2}{g}\frac{s^2}{(s+a)(s-a)} \\
&= -\frac{a^2}{g}\left(1+\frac{a^2}{s^2-a^2}\right) \\
&= -\frac{a^2}{g}\left(1+\frac{-a/2}{s+a}+\frac{a/2}{s-a}\right) \\
&= -\frac{a^2}{g}\delta(t)+\frac{a^3}{2g}e^{-at}-\frac{a^3}{2g}e^{at}
\end{aligned}
$$

It is unstable and will blow up.

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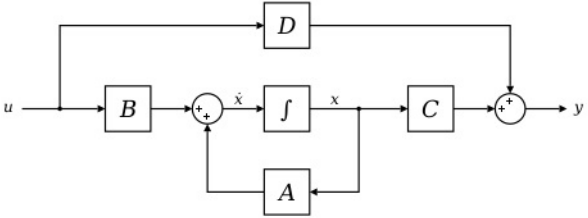

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

$$s\mathbf{X}(s) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s) \Rightarrow \mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s)$$
$$\mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(s) \Rightarrow \mathbf{Y}(s) = (\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D})\mathbf{U}$$

$$\mathbf{G}(s) = \frac{\mathbf{Y}(s)}{\mathbf{U}(s)} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

Recap: Matrix Inverse

In order to learn similarity transformation, we need review the definition of matrix inverse.
If matrix $\mathbf{A}$ is square, and (square) matrix $\mathbf{B}$ satisfies

$$\mathbf{BA} = \mathbf{AB} = \mathbf{I}$$

then $\mathbf{B}$ is called the inverse of A and is denoted as $\mathbf{B} = \mathbf{A}^{-1}$ For the inverse to exist, $\mathbf{A}$ must have a nonzero determinant, i.e., $\mathbf{A}$ must be non-singular. When this is true, $\mathbf{A}$ has a unique inverse given by

$$\mathbf{A}^{-1} = \frac{\mathbf{C}^T}{|\mathbf{A}|}$$

where $\mathbf{C}$ is the matrix formed by the cofactors C_{ij}. The matrix $\mathbf{C}^T$ is called the adjoint matrix, $\mathrm{Adj}(\mathbf{A})$. Thus the inverse of a nonsingular matrix is

$$\mathbf{A}^{-1} = \mathrm{Adj}(\mathbf{A})/|\mathbf{A}|$$

Example (Group Discussion)

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} x$$

$$(s\mathbf{I}-\mathbf{A}) = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 6 & 11 & s+6 \end{bmatrix} \to \Delta(s) = \det(s\mathbf{I}-\mathbf{A}) = s^3+6s^2+11s+6 = (s+1)(s+2)(s+3)$$

- Co-factor matrix: $\mathbf{C} = \begin{bmatrix} s^2+6s+11 & -6 & -6s \\ s+6 & s(s+6) & -11s-6 \\ 1 & s & s^2 \end{bmatrix}$
- Adjugate: $\mathrm{adj}(s\mathbf{I}-\mathbf{A}) = \mathbf{C}^T = \begin{bmatrix} s^2+6s+11 & s+6 & 1 \\ -6 & s(s+6) & s \\ -6s & -11s-6 & s^2 \end{bmatrix}$

Example

$$(s\mathbf{I}-\mathbf{A})^{-1}=\frac{1}{\Delta(s)}\,\mathrm{adj}(\mathbf{sI}-\mathbf{A})=\frac{1}{(s+1)(s+2)(s+3)}\begin{bmatrix} s^2+6s+11 & s+6 & 1 \\ -6 & s(s+6) & s \\ -6s & -11s-6 & s^2 \end{bmatrix}$$

$$\mathbf{G}(s)=\frac{\mathbf{Y}(s)}{\mathbf{U}(s)}=\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}+\mathbf{D}$$

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Definition of Realization

A realization of a transfer function $G(s)$ is any $\{A, B, C, D\}$ s.t.

$$G(s) = C(sI - A)^{-1}B + D$$

If the set exists, $G(s)$ is **realizable**

Condition of Realizability: SS $\rightarrow$ TF

Theorem

$G(s)$ is realizable $\Leftrightarrow$ each element of $G_{ij}(s)$ is a proper rational TF.

- A transfer function is said to be proper if its relative degree is greater than or equal to 0, and strictly proper if the relative degree is greater than or equal to 1.
- Let $G(s)$ have realization $\{A, B, C, D\}$. $(sI - A)^{-1} = \mathrm{adj}(sI - A)/\Delta(s)$
  Degree of $\mathrm{adj}(sI - A) \leq n - 1$ and degree of $\Delta(s) = n$
  $\Rightarrow (sI - A)^{-1}$ is a strictly proper rational TF matrix which is a vector space.
  $\Rightarrow C(sI - A)^{-1}B$ is also a strictly proper rational.
- If $D \neq \mathbf{0}$, G is proper rational; otherwise, G is strictly proper rational.
- Every TF with SS realization has proper/rational entries
- If the transfer function of the system is proper, then the system is causal. If the transfer function of a system has relative degree equal to 0 then there is also instantaneous transfer between input and output.

Example: the inverted pendulum is not strictly proper. It is a causal system.

Controllable Canonical Forms for the SISO System

Given a SISO system:

$$G(s) = \frac{b_n s^n + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}$$

the controllable canonical form is given by

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & \cdots & \cdots & -a_{n-1} \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} b_0 - b_n a_0 & b_1 - b_n a_1 & \cdots & b_{n-1} - b_n a_{n-1} \end{bmatrix} x + b_n u$$

Proof

Let z defined as $sz = Az + B$. $z = \begin{bmatrix} z_1 & z_2 & \dots & z_n \end{bmatrix}^T = (sI - A)^{-1}B$ with $z_i \in \mathbb{R}$.

From the form of A matrix

$$z_2 = sz_1, z_3 = sz_2 = s^2 z_1, \cdots, z_n = sz_{n-1} = s^{n-1} z_1$$

Now, from the last row of A,
$sz_n = -a_0 z_1 - a_1 z_2 - \cdots - a_{n-1} z_n + 1 = 1 - \sum_{i=0}^{n-1} a_i z_{i+1} = 1 - \sum_{i=0}^{n-1} a_i s^i z_1$
$s^n z_1 + \sum_{i=0}^{n-1} a_i s_i z_1 = 1 \Rightarrow z_1 = 1/\Delta(s) \Rightarrow z_{i+1} = s^i/\Delta(s)$
$G(s) = Cz + b_n = b_n + \sum_{i=0}^{n-1} (b_i - b_n a_i) z_{i+1} = b_n + \sum_{i=0}^{n-1} (b_i - b_n a_i) s^i/\Delta(s)$

$$= \frac{\sum_{i=0}^{n-1} b_i s^i + b_n(\Delta(s) - \sum_{i=0}^{n-1} a_i s^i)}{\Delta(s)} = \frac{\sum_{i=0}^{n} b_i s^i}{\Delta(s)} = \frac{b_n s^n + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}$$

Observable Canonical Form for the SISO System

Given a SISO system:

$$G(s) = \frac{b_n s^n + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}$$

can be converted into the analogous form

$$\dot{x} = \begin{bmatrix} -a_{n-1} & 1 & 0 & \cdots & 0 \\ \vdots & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \\ -a_1 & 0 & 0 & \cdots & 1 \\ -a_0 & 0 & \cdots & \cdots & 0 \end{bmatrix} x + \begin{bmatrix} b_{n-1} - b_n a_{n-1} \\ \vdots \\ b_1 - b_n a_1 \\ b_0 - b_n a_0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} x + b_n u$$

Similarity Transformations for Controllable/Observable Canonical Forms

- We can get here via a similarity transformation and an observable state space model.
- Let $M_c = PP_c^{-1}$, with P_c the controllability matrix for the system in controllable canonical form, calculated by A_c and B_c
- Let $M_o = Q^{-1}Q_o$, with Q_o the observability matrix for the system in observable canonical form, calculated by A_o and C_o

Q_o^{-1} and P_c^{-1} have special forms:

$$P_c^{-1} = \begin{bmatrix} a_1 & a_2 & \cdots & a_{n-1} & 1 \\ a_2 & a_3 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & 0 \\ a_{n-1} & 1 & \cdots & \ddots & \vdots \\ 1 & 0 & \cdots & \cdots & 0 \end{bmatrix}, \quad Q_o^{-1} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ a_{n-1} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_2 & a_3 & \cdots & 1 & 0 \\ a_1 & a_2 & \cdots & a_{n-1} & 1 \end{bmatrix}$$

These can be written by inspection!

Realization for MIMO Systems

Theorem

$G(s)$ is realizable $\Leftrightarrow$ each element of $G_{ij}(s)$ is a proper rational TF.

- Decompose G into a strictly proper TF matrix and a constant matrix:
  $G(s) = G_{sp}(s) + D \Rightarrow D = G(s = \infty)$
- Let $\Delta(s) = s^r + \alpha_1 s^{r-1} + \cdots \alpha_{r-1} s + \alpha_r$ be the least common denominator for $G_{sp}(s)$
  $\Rightarrow G_{sp} = \frac{1}{\Delta(s)}[N_1 s^{r-1} + N_2 s^{r-2} + \cdots + N_{r-1}s + N_r]$

$$\dot{x} = Ax + Bu = \begin{bmatrix} -\alpha_1 I_p & -\alpha_2 I_p & \cdots & -\alpha_{r-1} I_p & -\alpha_r I_p \\ I_p & 0_p & \cdots & 0_p & 0_p \\ 0_p & I_p & \cdots & 0_p & 0_p \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0_p & 0_p & \cdots & I_p & 0_p \end{bmatrix} x + \begin{bmatrix} I_p \\ 0_p \\ 0_p \\ \vdots \\ 0_p \end{bmatrix} u,$$

$$y = Cx + Gu = \begin{bmatrix} N_1 & N_2 & \cdots & N_{r-1} & N_r \end{bmatrix} x + G(\infty)u$$

is a realization for a $q \times p$ TF matrix G. $u \in \mathbb{R}^{p\times 1}$, $x \in \mathbb{R}^{rp\times 1}$, $y \in \mathbb{R}^{q\times 1}$, $N_i \in \mathbb{R}^{q\times p}$

Example

- Realize

$$G(s) = \begin{bmatrix} \frac{4s-10}{2s+1} & \frac{3}{s+2} \\ \frac{1}{(2s+1)(s+2)} & \frac{s+1}{(s+2)^2} \end{bmatrix}$$

- Solution:
  Find G_{sp}:

$$G(\infty) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow G_{sp} = \begin{bmatrix} \frac{-6}{s+0.5} & \frac{3}{s+2} \\ \frac{0.5}{(s+0.5)(s+2)} & \frac{s+1}{(s+2)^2} \end{bmatrix}$$

$$d(s) = (s+0.5)(s+2)^2 = s^3 + 4.5s^2 + 6s + 2$$

$$\Rightarrow G_{sp} = \frac{1}{s^3+4.5s^2+6s+2} \begin{bmatrix} -6(s+2)^2 & 3(s+2)(s+0.5) \\ 0.5(s+2) & (s+0.5)(s+1) \end{bmatrix}$$

Example

$$\Rightarrow N_1(s) = \begin{bmatrix} -6 & 3 \\ 0 & 1 \end{bmatrix}, N_2(s) = \begin{bmatrix} -24 & 7.5 \\ 0.5 & 1.5 \end{bmatrix}, N_3(s) = \begin{bmatrix} -24 & 3 \\ 1 & 0.5 \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} -4.5 & 0 & -6 & 0 & -2 & 0 \\ 0 & -4.5 & 0 & -6 & 0 & -2 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} -6 & 3 & -24 & 7.5 & -24 & 3 \\ 0 & 1 & 0.5 & 1.5 & 1 & 0.5 \end{bmatrix}, D = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

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Controllable Decomposition

One way to reduce the redundant states in realization is to use Kalman decomposition. We will again use Similarity Transformation. Let $x = M\hat{x}$

- Form the change of basis matrix M whose first $n_c = r(P)$ columns are linearly independent columns of P and last $n - n_c$ columns are arbitrary s.t. $r(M) = n$

$$\Rightarrow \hat{A} = M^{-1}AM = \begin{bmatrix} A_c & A_{12} \\ 0 & A_{\bar{c}} \end{bmatrix}, \hat{B} = M^{-1}B = \begin{bmatrix} B_c \\ 0 \end{bmatrix}, \hat{C} = CM = \begin{bmatrix} C_c & C_{\bar{c}} \end{bmatrix}$$

$$\Rightarrow \dot{\hat{x}}_c = \begin{bmatrix} A_c & A_{12} \\ 0 & A_{\bar{c}} \end{bmatrix} \begin{bmatrix} \hat{x}_c \\ \hat{x}_{\bar{c}} \end{bmatrix} + \begin{bmatrix} B_c \\ 0 \end{bmatrix} u, y = \begin{bmatrix} C_c & C_{\bar{c}} \end{bmatrix} \begin{bmatrix} \hat{x}_c \\ \hat{x}_{\bar{c}} \end{bmatrix} + Du$$

with $A_c \in \mathbb{R}^{n_c \times n_c}$, $A_{\bar{c}} \in \mathbb{R}^{(n-n_c)\times(n-n_c)}$

$\begin{aligned} \dot{\hat{x}}_c &= A_c\hat{x}_c + B_c u \\ y &= C_c\hat{x}_c + Du \end{aligned}$ is controllable and has the same TF as $\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$

Transfer Function Claim

$$
\begin{aligned}
G &= C(sI - A)^{-1}B + D \\
&= \begin{bmatrix} C_c & C_{\bar{c}} \end{bmatrix} \begin{bmatrix} sI - A_c & -A_{12} \\ 0 & sI - A_{\bar{c}} \end{bmatrix}^{-1} \begin{bmatrix} B_c \\ 0 \end{bmatrix} + D \\
&= \begin{bmatrix} C_c & C_{\bar{c}} \end{bmatrix} \begin{bmatrix} (sI - A_c)^{-1} & (sI - A_c)^{-1}A_{12}(sI - A_{\bar{c}})^{-1} \\ 0 & (sI - A_{\bar{c}})^{-1} \end{bmatrix} \begin{bmatrix} B_c \\ 0 \end{bmatrix} + D \\
&= C_c(sI - A_c)^{-1}B_c + D
\end{aligned}
$$

Observable Decomposition

We can similarly break into observable, unobservable parts by using M^{-1} whose first n_o rows are linearly independent rows of Q and last $n - n_o$ rows are arbitrary s.t. $r\left(M^{-1}\right) = n$

$$\Rightarrow \begin{bmatrix} \dot{\hat{x}}_o \\ \hat{x}_{\overline{o}} \end{bmatrix} = \begin{bmatrix} A_o & 0 \\ A_{21} & A_{\overline{o}} \end{bmatrix} \begin{bmatrix} \hat{x}_o \\ \hat{x}_{\overline{o}} \end{bmatrix} + \begin{bmatrix} B_o \\ B_{\overline{o}} \end{bmatrix} u,\ y = \begin{bmatrix} C_o & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_o \\ \hat{x}_{\overline{o}} \end{bmatrix} + Du$$

Furthermore, $\begin{array}{l} \dot{\hat{x}}_o = A_o \hat{x}_o + B_o u \\ y = C_o \hat{x}_o + Du \end{array}$ is observable and has the same TF as the full system.

Full Decomposition

By applying the controllability decomposition and observability decomposition of each subsystems, we can arrive at the following:

$$\begin{bmatrix} \dot{\hat{x}}_{co} \\ \dot{\hat{x}}_{c\bar{o}} \\ \dot{\hat{x}}_{\bar{c}o} \\ \dot{\hat{x}}_{\overline{co}} \end{bmatrix} = \begin{bmatrix} A_{co} & 0 & A_{13} & 0 \\ A_{21} & A_{c\bar{o}} & A_{23} & A_{24} \\ 0 & 0 & A_{\bar{c}o} & 0 \\ 0 & 0 & A_{43} & A_{\overline{co}} \end{bmatrix} \begin{bmatrix} \hat{x}_{co} \\ \hat{x}_{c\bar{o}} \\ \hat{x}_{\bar{c}o} \\ \hat{x}_{\overline{co}} \end{bmatrix} + \begin{bmatrix} B_{co} \\ B_{c\bar{o}} \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} C_{co} & 0 & C_{\bar{c}o} & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_{co} \\ \hat{x}_{c\bar{o}} \\ \hat{x}_{\bar{c}o} \\ \hat{x}_{\overline{co}} \end{bmatrix} + Du$$

Furthermore, $\begin{array}{l} \dot{\hat{x}}_{co} = A_{co}\hat{x}_{co} + B_{co}u \\ y = C_{co}\hat{x}_{co} + Du \end{array}$ is controllable and observable and has the same TF as the full system.

Full Decomposition

There exists a coordinate transformation $\hat{x} = M^{-1}x \in \mathbb{R}^n$ such that

$$\dot{\hat{x}} = \begin{bmatrix} A_{co} & 0 & A_{13} & 0 \\ A_{21} & A_{c\bar{o}} & A_{23} & A_{24} \\ 0 & 0 & A_{\bar{c}o} & 0 \\ 0 & 0 & A_{43} & A_{\overline{co}} \end{bmatrix} \hat{x} + \begin{bmatrix} B_{co} \\ B_{c\bar{o}} \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} C_{co} & 0 & C_{\bar{c}o} & 0 \end{bmatrix} \hat{x} + Du$$

with $M = [m_1 \ m_2 \ m_3 \ m_4]$ where

- $m_2 \to (c\bar{o}) : Range(P) \cap Null(Q) \Rightarrow$ identify m_2
- $m_1 \to (co) : Range(P) = m_1 \cup m_2 \Rightarrow$ identify m_1 by subtracting m_2 from $Range(P)$
- $m_4 \to (\overline{co}) : Null(Q) = m_2 \cup m_4 \Rightarrow$ identify m_4 by subtracting m_2 from $Null(Q)$
- $m_3 \to (\bar{c}o) : m_1 \cup m_2 \cup m_3 \cup m_4 =$ basis for $\mathbb{R}^n \Rightarrow$ identify m_3 by subtracting m_1, m_2, m_4 from $\mathbb{R}^n$

Example of Kalman Decomposition

$\dot{x} = \begin{bmatrix} 2 & 1 & 1 \\ 5 & 3 & 6 \\ -5 & -1 & -4 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u, y = \begin{bmatrix} 1 & 1 & 2 \end{bmatrix} x$. Do the Kalman full decomposition.

$$\Rightarrow P = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 5 & -5 \\ 0 & -5 & 5 \end{bmatrix}$$

$$\Rightarrow r(P) = 2 \Rightarrow M = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 5 & 0 \\ 0 & -5 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \dot{\hat{x}}_c \\ \dot{\hat{x}}_{\bar{c}} \end{bmatrix} = \begin{bmatrix} 0 & 6 & -1.4 \\ 1 & -1 & 1.2 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} \hat{x}_c \\ \hat{x}_{\bar{c}} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u, \; y = \begin{bmatrix} 1 & -3 & 2 \end{bmatrix} \begin{bmatrix} \hat{x}_c \\ \hat{x}_{\bar{c}} \end{bmatrix}$$

$Q = \begin{bmatrix} 1 & 1 & 2 \\ -3 & 2 & -1 \\ 9 & 4 & 13 \end{bmatrix} \Rightarrow r(Q) = 2 \Rightarrow$ unobservable

Example of Kalman Decomposition cont.

Let's look at same test for controllable subsystem

$A_c = \begin{bmatrix} 0 & 6 \\ 1 & -1 \end{bmatrix}, C_c = \begin{bmatrix} 1 & -3 \end{bmatrix} \Rightarrow Q_c = \begin{bmatrix} 1 & -3 \\ -3 & 9 \end{bmatrix} \Rightarrow r(Q_c) = 1 \Rightarrow$

System has 1 co, 1 $c\bar{o}$, 1 $\bar{c}o$ mode.

Let $M^{-1} = \begin{bmatrix} 1 & -3 \\ 1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} \dot{\hat{x}}_{co} \\ \dot{\hat{x}}_{c\bar{o}} \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} \hat{x}_{co} \\ \hat{x}_{c\bar{o}} \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u,\ y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_{co} \\ \hat{x}_{c\bar{o}} \end{bmatrix}$

$\Rightarrow \dot{\hat{x}}_{co} = -3\hat{x}_{co} + u,\ \ y = \hat{x}_{co}$

This is the minimal realization of

$$\Rightarrow G(s) = \frac{1}{s+3}$$

Directly Using Full Decomposition

$$P = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 5 & -5 \\ 0 & -5 & 5 \end{bmatrix}, Q = \begin{bmatrix} 1 & 1 & 2 \\ -3 & 2 & -1 \\ 9 & 4 & 13 \end{bmatrix}$$

$$\mathscr{R}(P) = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ -5 \end{bmatrix} \right) = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right)$$

Use Gaussian Elimination to get $\mathscr{N}(Q)$: $\begin{bmatrix} 1 & 1 & 2 & 0 \\ -3 & 2 & -1 & 0 \\ 9 & 4 & 13 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 5 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow$

$$\begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \mathscr{N}(Q) = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right)$$

Directly Using Full Decomposition

- $m_2 \to (c\bar{o})$: $Range(P) \cap Null(Q) \Rightarrow$ identify m_2
- $m_1 \to (co)$: $Range(P) = m_1 \cup m_2 \Rightarrow$ identify m_1 by subtracting m_2 from $Range(P)$
- $m_4 \to (\overline{co})$: $Null(Q) = m_2 \cup m_4 \Rightarrow$ identify m_4 by subtracting m_2 from $Null(Q)$
- $m_3 \to (\bar{c}o)$: $m_1 \cup m_2 \cup m_3 \cup m_4 =$ basis for $\mathbb{R}^n \Rightarrow$ identify m_3 by subtracting m_1, m_2, m_4 from $\mathbb{R}^n$

$$\mathscr{R}(P) = \text{span}\left(\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\-1\end{bmatrix}\right), \mathscr{N}(Q) = \text{span}\left(\begin{bmatrix}1\\1\\-1\end{bmatrix}\right)$$

$$m_2 = \begin{bmatrix}1\\1\\-1\end{bmatrix}, m_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, m_4 = \emptyset, m_3 = \begin{bmatrix}0\\1\\0\end{bmatrix}, M = \begin{bmatrix}1 & 1 & 0\\0 & 1 & 1\\0 & -1 & 0\end{bmatrix}, M^{-1} = \begin{bmatrix}1 & 0 & 1\\0 & 0 & -1\\0 & 1 & 1\end{bmatrix}$$

$$\hat{A} = M^{-1}AM = \begin{bmatrix}-3 & 0 & 0\\5 & 2 & 1\\0 & 0 & 2\end{bmatrix}, \hat{B} = M^{-1}B = \begin{bmatrix}1\\0\\0\end{bmatrix}, \hat{C} = CM = [1, 0, 1]$$

$$\Rightarrow \dot{\hat{x}}_{co} = -3\hat{x}_{co} + u,\ y = \hat{x}_{co}$$

Table of Contents

Minimal Realization

Minimal realization: the realization with the minimum number of states.

- Different realizations of a TF can have a varying number of states
- We have already seen with the Kalman decomposition that the uncontrollable/unobservable states do not affect the TF
- The minimal realization is given for the KD by

$$\dot{x}_{co} = A_{co}x_{co} + B_{co}u$$
$$y = C_{co}x_{co} + Du$$

Minimal Realization

Theorem

A realization is minimal $\Leftrightarrow$ it is controllable & observable.

Theorem

All minimal realizations of the same transfer function are similar