Module 1-5: Stability

Linear Control Systems (2020)

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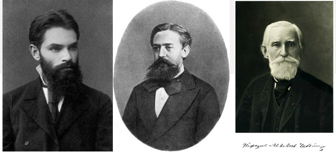
Stability

Lyapunov 1857-1918
Markov 1856-1922
Chebyshev 1821-1894

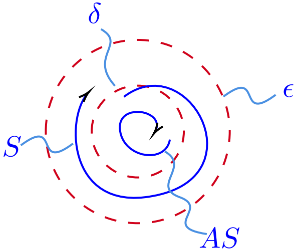


Stability is one fundamental dynamic property of a system. Essentially we care about

Does the solution behave well as $t \to \infty$?

Stability only depends on the zero input response at equilibrium points

- CT systems: $\dot{x} = f(\overline{x}, t_0, t) = 0$
- DT systems: $x[k+1] = x[k]$



Examples: balancing stones, Tacoma Narrows Bridge, biped robots, spinning of cars.

Definitions of Stability in the Sense of Lyapunov (i.s.L)

There are various ways to define "well-behaved".

- An equilibrium point $\overline{x}$ of $\dot{x} = A(t)x$ is **stable i.s.L** if, $\forall \epsilon > 0, \exists \delta(t_0, \epsilon) > 0$ s.t. $||x(t_0) - \overline{x}|| < \delta, ||x(t) - \overline{x}|| < \epsilon, \forall t > t_0$.
- If $\delta = \delta(\epsilon)$ (independent of t_0), $\overline{x}$ is **uniformly stable** (time invariant).
- If $||x(t) - \overline{x}|| \to 0$ as $t \to \infty$, $\overline{x}$ is **asymptotically stable**.
- If $\overline{x}$ is asymptotically stable and $\exists \delta > 0, \gamma > 0, \lambda > 0$ s.t. $||x(t) - \overline{x}|| \leq \gamma e^{-\lambda t}||x(t_0) - \overline{x}||$, $\overline{x}$ is **exponentially stable**.

Recap: Example

$$\dot{\mathbf{x}} = \begin{bmatrix} x_1 - x_1^3 + x_1 x_2 \\ -x_2 + 2x_1 x_2 \end{bmatrix}$$

1. Find equilibria

$\begin{cases} x_1 - x_1^3 + x_1 x_2 = 0 \\ -x_2 + 2x_1 x_2 = 0 \end{cases} \Rightarrow x_2(2x_1 - 1) = 0 \Rightarrow x_2 = 0$ or $x_1 = \frac{1}{2}$

$x_2 = 0 \Rightarrow x_1(1 - x_1^2) = 0 \Rightarrow x_1 = 0, x_1 = 1, x_1 = -1 \Rightarrow (0,0), (1,0), (-1,0)$

$x_1 = \frac{1}{2} \Rightarrow \frac{1}{2} - \frac{1}{2^3} + \frac{1}{2}x_2 = 0 \Rightarrow x_2 = (-1 + \frac{1}{2^2}) \Rightarrow (\frac{1}{2}, -\frac{3}{4})$ 2. Linearization

$$\frac{\partial f}{\partial \mathbf{x}} = \begin{bmatrix} 1 - 3x_1^2 + x_2 & x_1 \\ 2x_2 & 2x_1 - 1 \end{bmatrix}$$

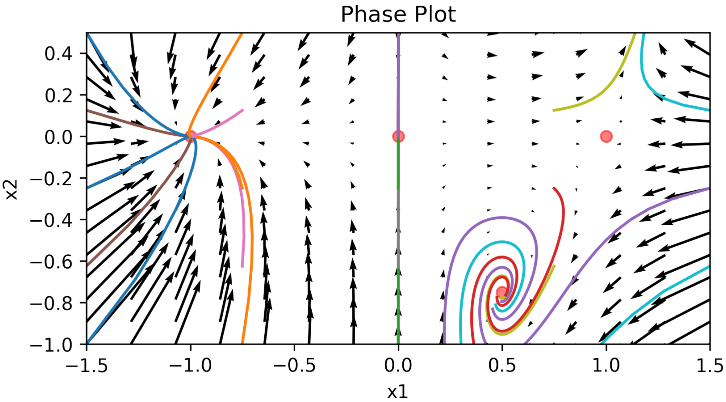
Recap: Phase Portrait Plot

$(0,0)$: $\dot{\delta}_{\mathbf{x}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \delta_{\mathbf{x}}$

$(-1,0)$: $\dot{\delta}_{\mathbf{x}} = \begin{bmatrix} -2 & -1 \\ 0 & -3 \end{bmatrix} \delta_{\mathbf{x}}$

$(1,0)$: $\dot{\delta}_{\mathbf{x}} = \begin{bmatrix} -2 & 1 \\ 0 & 1 \end{bmatrix} \delta_{\mathbf{x}}$

$(\frac{1}{2}, -\frac{3}{4})$: $\dot{\delta}_{\mathbf{x}} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & 0 \end{bmatrix} \delta_{\mathbf{x}}$



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Stability of CT LTI Systems

Theorem

$\bar{x}$ for $\dot{x} = Ax$ is stable $\Leftrightarrow$

all e-values of A have non-positive real parts, and those with zero real parts are non-defective.

Proof:

For LTI systems, the solution to zero input response is $x(t) = e^{At}x(0)$. $x(0) < \infty$. If elements of e^{At} are finite as $t \to \infty$, then the system is stable i.s.L.

Recap: Ways to compute e^{At}

1. Apply the series definition:
   $e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}$
2. Apply Cayley-Hamilton theorm:
   $e^{At} = \beta_0 I + \beta_1 A + \cdots + \beta_{n-1} A^{n-1}$
3. Use similarity transformations:
   $e^{At} = M e^{Jt} M^{-1}$
4. [New] Inverse Laplace Transformation:
   Compute $(sI - A)^{-1}$, then compute $\mathscr{L}^{-1}\{i, j^{th}$ element of $(sI - A)^{-1}\}$. Gives i, j^{th} element of e^{At}.
   $\dot{x} = Ax + Iu$. Assume $x(0) = 0$, $\Rightarrow sX(s) = AX(s) + IU(s) \Rightarrow X(s) = (sI - A)^{-1} U(s)$
   If $u(t) = \delta(t)$, $U(s) = 1$. $x(t) = \mathscr{L}^{-1}((sI - A)^{-1})$
   $x(t) = e^{At} x(0) + \int_0^t e^{A(t-\tau)} I \delta(\tau) d\tau = e^{At} \int_0^t e^{-\tau} \delta(\tau) d\tau =$
   $e^{At} (\int_0^{0^+} e^{-\tau} \delta(\tau) d\tau + \int_{0^+}^t e^{-\tau} \delta(\tau) d\tau) = e^{At} \int_0^{0^+} 1 \delta(\tau) d\tau = e^{At}$

Which one may help to analyze the stability?

Recap: Similarity Decomposition

Using a similarity transformation, we can convert the state equation into diagonal (or Jordan) form.

Let $x = M\mathbf{x}$ where $M = [v_1 \vdots v_2 \vdots \cdots \vdots v_n]$ are the eigenvectors (or generated e-vectors) of A

$$\begin{cases} \dot{\mathbf{x}} = M^{-1}AM\mathbf{x} + M^{-1}Bu \\ y = CM\mathbf{x} + Du \end{cases}$$

Here $J = M^{-1}AM$ is in either diagonal or Jordan form. In either case, e^{Jt} is easier to computer.

Recap: Jordan Decomposition In General

Let $\mathbf{A}$ be an $n \times n$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ of algebraic multiplicities $m_1, ..., m_p$ and geometric multiplicities $q_1, ..., q_p$. Then $\exists$ an invertible matrix $\mathbf{M}$ such that $\mathbf{J} = \mathbf{M}^{-1}\mathbf{A}\mathbf{M}$, where

$$\mathbf{J} = \begin{bmatrix} \hat{\mathbf{J}}_1 & 0 & 0 & 0 \\ 0 & \hat{\mathbf{J}}_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \hat{\mathbf{J}}_p \end{bmatrix}_{n\times n}$$

$\#$blocks $= p(\#$distinct e-values$)$

$$\hat{\mathbf{J}}_i = \begin{bmatrix} \hat{\mathbf{J}}_{i1} & 0 & 0 & 0 \\ 0 & \hat{\mathbf{J}}_{i2} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \hat{\mathbf{J}}_{iq_i} \end{bmatrix}_{m_i\times m_i} \qquad \hat{\mathbf{J}}_{ij} = \begin{bmatrix} \lambda_i & 1 & 0 \\ 0 & \ddots & 1 \\ 0 & 0 & \lambda_i \end{bmatrix}_{?\times ?, ?\geq 2}$$

$\#$blocks $= q_i($ $\#$indep e-vectors assoc. with $\lambda_i)$ In general, we do not know what is the dimensions for the 3rd level Jordan blocks except in type I, II_1, or II_2.

Recap: Exponential of Jordan Form cont.

Given $J = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, find e^{tJ}

$$J = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} \Rightarrow e^{tJ} = \begin{bmatrix} e^{tJ_1} & 0 \\ 0 & e^{tJ_2} \end{bmatrix}$$

$$J_1 = D + N = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$e^{tD} = \begin{pmatrix} e^{2t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{2t} \end{pmatrix}$. Need to compute e^{tN}

Recap: Exponential of Jordan Form

Apply Cayley-Hamilton theorem:

$$\lambda = 0, f(N) = e^{tN}, N^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$f(\lambda) = e^{t\lambda} = 1, f'(\lambda) = te^{t\lambda} = t, f''(\lambda) = t^2 e^{t\lambda} = t^2$$

$$g(\lambda) = \beta_2 \lambda^2 + \beta_1 \lambda + \beta_0 = \beta_0, g'(\lambda) = 2\beta_2 \lambda + \beta_1 = \beta_1, g''(\lambda) = 2\beta_2$$

$$f(\lambda) = g(\lambda), f'(\lambda) = g'(\lambda), f''(\lambda) = g''(\lambda) \Rightarrow \beta_0 = 1, \beta_1 = t, \beta_2 = \tfrac{1}{2}t^2$$

$$e^{tN} = f(N) = g(N) = \tfrac{1}{2}t^2 N^2 + tN + I = \begin{bmatrix} 1 & t & \frac{1}{2}t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$$

Finally, because $DN = ND$, $e^{tJ_1} = e^{t(D+N)} = e^{tD} \cdot e^{tN} = \begin{bmatrix} e^{2t} & te^{2t} & \frac{1}{2}t^2 e^{2t} \\ 0 & e^{2t} & te^{2t} \\ 0 & 0 & e^{2t} \end{bmatrix}$

Recap: Exponential of Jordan Form

$$e^{Jt} = \begin{bmatrix} e^{\lambda_1 t} & 0 & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & te^{\lambda_2 t} & \cdots & 0 \\ 0 & 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \end{bmatrix}$$

e^{Jt} has terms of the form $t^m e^{\lambda_i t}$, with $m \neq 0$ for Jordan blocks of order > 1. We will use this trick again in the stability analysis.

CT LTI Systems

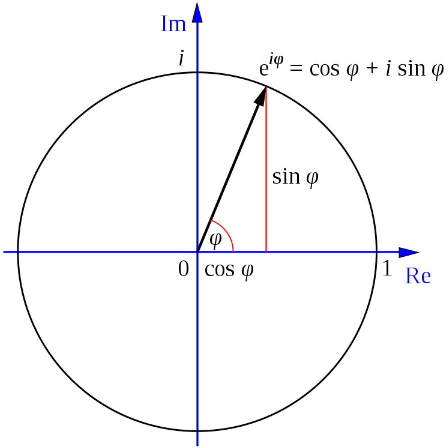
$$e^{Jt} = \begin{bmatrix} e^{\lambda_1 t} & 0 & 0 & \cdots & \cdots & 0 \\ 0 & e^{\lambda_2 t} & te^{\lambda_2 t} & t^2 e^{\lambda_2 t} & \cdots & 0 \\ 0 & 0 & e^{\lambda_2 t} & te^{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots & \end{bmatrix}$$

e^{Jt} has terms of the form $t^m e^{\lambda_i t}$, with $m \neq 0$ for Jordan blocks of order > 1. We just need to make sure

$$t^m e^{\lambda_i t} \text{ all bounded}$$

How to check $e^{\lambda_i t}$? Remember λ is a complex number.

Euler's Formula



Euler, 1707-1783

CT LTI Systems

$$e^{Jt} = \begin{bmatrix} e^{\lambda_1 t} & 0 & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & te^{\lambda_2 t} & \cdots & 0 \\ 0 & 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \end{bmatrix}$$

e^{Jt} has terms of the form $t^m e^{\lambda_i t}$, with $m \neq 0$ for Jordan blocks of order > 1. We just need to make sure

$$t^m e^{\lambda_i t} \text{ all bounded}$$

Let $\lambda_i = R_e + I_m j$, $t^m e^{\lambda_i t}$ can be written as

$$t^m e^{R_e t}(\cos(I_m t) + j \sin(I_m t))$$

Stability of CT LTI Systems

Given

$$t^m e^{R_e t}(\cos(I_m t) + j\sin(I_m t)),$$

consider the following cases:

1. $\forall \lambda_i \ , R_e < 0 \quad \Rightarrow$ Asymptotic stable
2. $\exists \lambda_i \ , R_e > 0 \quad \Rightarrow$ Unstable
3. $\exists \lambda_i \ , R_e = 0 \ , m = 0 \quad \Rightarrow$Stable i.s.L.
4. $\exists \lambda_i \ , R_e = 0 \ , m > 0 \ \Rightarrow$ Unstable

Stability of DT LTI

Theorem

$\overline{x} = 0$ *for* $x(k+1) = Ax(k)$ *is stable* $\Leftrightarrow$

all eigenvalues of A *satisfy* $|\lambda_i| \leqslant 1$ *and all* λ_i*=1 are non-defective*

.

Stability of DT LTI

$\dot{x}(k) = A^k x(0) = MJ^kM^{-1} = x(0)$ with J in Jordan form

$$J^k = \begin{bmatrix} \lambda_1^k & k\lambda_1^{k-1} & 0 & 0 \\ 0 & \lambda_1^k & 0 & 0 \\ 0 & \dots & \lambda_2^k & \dots \\ 0 & \dots & \dots & \dots \end{bmatrix}$$

Use programming to check it: [COLAB LINK]
The system is stable $\Leftrightarrow k^m\lambda^k$ is bounded as $k \to \infty$.
Write $\lambda_i = r_i e^{j\theta_i}$. The system is stable $\Leftrightarrow k^m r_i^k e^{j\theta_i k} = k^m r_i^k(\cos(\theta_i k) + j\sin(\theta_i k))$ is bounded as $k \to \infty$.

Stability of DT LTI

Consider $k^m r_i^k e^{j\theta_i k} = k^m r_i^k(\cos(\theta_i k) + j\sin(\theta_i k))$

1. $\forall \lambda_i, r_i < 1 \Rightarrow$ Asymptotic Stable
2. $\exists \lambda_i, r_i > 1 \Rightarrow$ Unstable
3. $\exists \lambda_i, r_i = 1 \;\&\; m = 0 \Rightarrow$ Stable i.s.L.
4. $\exists \lambda_i, r_i = 1 \;\&\; m > 0 \Rightarrow$ Unstable

Asymptotic Stability of LTI Systems

$\bar{x} = 0$ for $\dot{x} = Ax$ is asymptotically stable $\Leftrightarrow$ all eigenvalues have negative real parts. $\bar{x} = 0$ for $x(k+1) = Ax(k)$ is $AS \Leftrightarrow$ all eigenvalues of A satisfy $|\lambda_i| < 1$.

Every asymptotically stable LTI system is exponentially stable.

Why?

This follows directly from case from the prior equations.

$$\begin{cases} \lim_{x\to\infty} t^m e^{R_e t}[\cos(I_m t + j\sin(I_m t)] = 0 \Leftrightarrow u < 0 \\ \lim_{k\to\infty} (k)^m \lambda_i^k = 0 \Leftrightarrow \lambda_i < 1 \end{cases}$$

Recap: Example

$$\dot{\mathbf{x}} = \begin{bmatrix} x_1 - x_1^3 + x_1 x_2 \\ -x_2 + 2x_1 x_2 \end{bmatrix}$$

1. Find equilibria

$$\begin{cases} x_1 - x_1^3 + x_1 x_2 = 0 \\ -x_2 + 2x_1 x_2 = 0 \end{cases} \Rightarrow x_2(2x_1 - 1) = 0 \Rightarrow x_2 = 0 \text{ or } x_1 = \frac{1}{2}$$

$x_2 = 0 \Rightarrow x_1(1 - x_1^2) = 0 \Rightarrow x_1 = 0, x_1 = 1, x_1 = -1 \Rightarrow (0,0), (1,0), (-1,0)$

$x_1 = \frac{1}{2} \Rightarrow \frac{1}{2} - \frac{1}{2^3} + \frac{1}{2}x_2 = 0 \Rightarrow x_2 = (-1 + \frac{1}{2^2}) \Rightarrow (\frac{1}{2}, -\frac{3}{4})$ 2. Linearization

$$\frac{\partial f}{\partial \mathbf{x}} = \begin{bmatrix} 1 - 3x_1^2 + x_2 & x_1 \\ 2x_2 & 2x_1 - 1 \end{bmatrix}$$

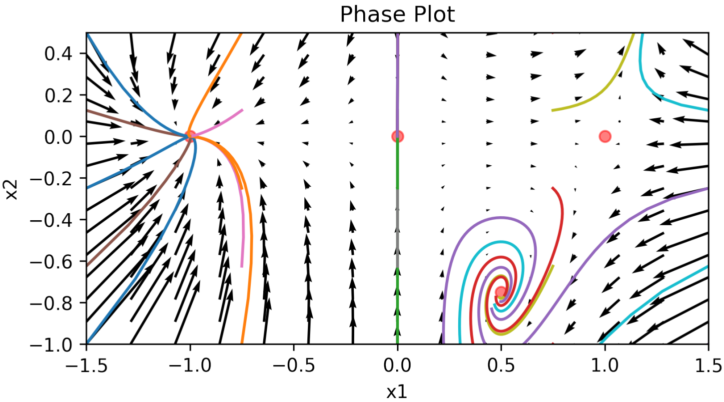
Phase Portrait Plot

$(0,0)$: $\dot{\delta}_{\mathbf{x}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \delta_{\mathbf{x}}$

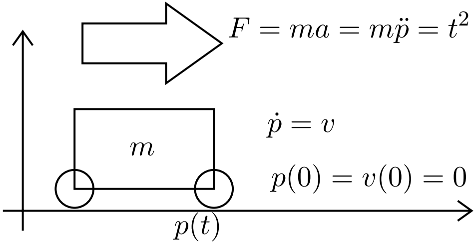
$(-1,0)$: $\dot{\delta}_{\mathbf{x}} = \begin{bmatrix} -2 & -1 \\ 0 & -3 \end{bmatrix} \delta_{\mathbf{x}}$

$(1,0)$: $\dot{\delta}_{\mathbf{x}} = \begin{bmatrix} -2 & 1 \\ 0 & 1 \end{bmatrix} \delta_{\mathbf{x}}$

$(\frac{1}{2}, -\frac{3}{4})$: $\dot{\delta}_{\mathbf{x}} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & 0 \end{bmatrix} \delta_{\mathbf{x}}$



Example: Driving



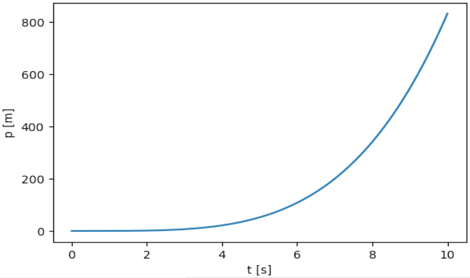
Which is the state?:
$\{p, \dot{p}, \ddot{p}\}$, $\{\dot{p}, \ddot{p}\}$, $\{p, \dot{p}\}$, $\{p\}$, solve $p(t)$

Is it stable? How about the discrete version of the system?

$x_1 = p$, $x_2 = \dot{p}$, $\mathbf{x} = [x_1, x_2]^T$

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} F$$

$$\mathbf{y} = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}$$



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The Eigenvalues Tests Cannot be Used on LTV Systems Directly

1. The eigenvalues of $A(t)$ at any instant t do not determine stability.
2. If the eigenvalues of $A(t) + A^T(t)$ are always negative, the system is asymptotically stable.
3. If all eigenvalues of $A(t) + A^T(t)$ are always positive, the system is unstable.
4. If all eigenvalues of $A(t)$ have negative real parts & $\exists V < \infty$ s.t. $||\dot{A}(t)|| < V$, the system is stable. (slowly time varying)

Note: We will not prove these claims.

Example

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-1 & e^{2t}\\ 0 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}, x_1(0) = 1, x_2(0) = 2$$

x_2 can be solved directly $\Rightarrow x_2(t) = e^{-t}x_2(0)$

$$x_1(t) = e^{-t}x_1(0) + \int_0^t e^{-(t-\tau)}x_2(0) \times e^{\tau}d\tau$$
$$= e^{-t}x_1(0) + x_2(0) \times e^{-t}\int_0^t e^{2\tau}d\tau$$
$$= e^{-t}x_1(0) + x_2(0) \times e^{-t} \times \frac{1}{2} \times e^{2\tau}|_0^t$$
$$= e^{-t}x_1(0) + \frac{x_2(0)}{2} \times (e^t - e^{-t})$$
$$= e^{-t} \times \left(x_1(0) - \frac{x_2(0)}{2}\right) + \frac{x_2(0)}{2} \cdot e^t = e^t$$

$\Rightarrow$ unstable, even though $\lambda_1 = \lambda_2 = -1$ are negative $\forall$ t

- Not slowly time varying

Stabilizability & Detectability

Stabilizability:

A system is stabilizable$\Leftrightarrow$its uncontrollable modes are Lyapunov stable.

Can use control to stabilize any unstable controllable modes.

Detectability:

A system is detectable $\Leftrightarrow$its unobservable modes are Lyapunov stable.

Note: Kalman Decomposition is useful. But blindly applying K-D is risky. We may hide the unstable states.

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How about the stability of the nonlinear system?

Linearization.

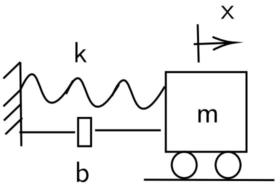
Lyapunov's Indirect (1^{st}) Method

Let $\dot{x} = f(x)$. Linearize the system, we have

- The origin is locally Asymptotically Stable if $Re(\lambda_i) < 0, \forall \lambda_i$ of A
- Unstable if $Re(\lambda_i) > 0$ for any λ_i.
  The implication is that we can design controllers for the linearized model & apply them to the original nonlinear system.
- What if $Re(\lambda_i) = 0 \Rightarrow$ very risky as we have used approximation for linearization (Taylor expansion).

Lyapunov's Direct (2^{nd}) Method - Illustrative Example

-Define an abstract "energy-like" quantity & show that it decreases along the system trajectories $\Rightarrow$ stable.



$\Rightarrow m\ddot{x} = -kx - b\dot{x}$

$$\dot{X} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} X, \; X = [x, \dot{x}]^T$$

The energy in the system is $V(x, \dot{x}) = \frac{1}{2}(m\dot{x}^2 + kx^2)$

- Now look at how the energy changes over time

$\dot{V}(x, \dot{x}) = m\dot{x}\ddot{x} + kx\dot{x} = \dot{x}(-b\dot{x} - kx) + kx\dot{x} = -b\dot{x}^2$

$\Rightarrow$ the energy decreases when the system has any positive velocity

$\Rightarrow$ the system must stop

- We now generalize this concept of energy to "Lyapunov functions"

Positive Definite Functions

- A function $V(x)$ is positive (negative) definite in a neighborhood of the origin if $V(x) > 0$ ($V(x) < 0$) for all $x \neq 0$ and $x(0) = 0$
- A function $V(x)$ is positive (negative) semidefinite in a neighborhood of the origin if $V(x) \geq 0$ ($V(x) \leq 0$) for all $x \neq 0$ and $x(0) = 0$

Theorem

The origin of $\dot{x} = f(x)$ is stable if

- $V(x)$ *and its partial derivatives are continuous*
- $V(x)$ *is positive definite*
- $\dot{V}(x)$ *is negative semidefinite*

If $\dot{V}(x)$ is negative definite $\exists V(x) > 0$, then the origin is asymptotically stable.

Example

Decide the stability of the following system

- $\dot{x}_1 = -x_1 - 2x_2^2$
- $\dot{x}_2 = x_1 x_2 - x_2^3$

Using $V(x) = \frac{1}{2}x_1^2 + x_2^2$
Clearly $V(x) > 0, \forall x \neq 0$

$\dot{V}(x) = x_1\dot{x_1} + 2x_2\dot{x}_2 = x_1(-x_1 - 2x_2^2) + 2x_2(x_1x_2 - x_2^3) = -x_1^2 - 2x_2^4 = -x_1^2 - 2x_2^4 < 0, \forall x \neq 0$

Note $V(x) \to \infty$ as $||x|| \to \infty \Rightarrow$ The origin is globally, uniformly asymptotically Stable.

DT Systems

All remains the same, except instead of $\dot{V}(x,t)$ we consider

$$\triangle V(x,k) = V(x(k+1)) - V(x(k))$$

Lyapunov's Direct (2^{nd}) Method for LTI Systems

Consider $V(x) = x^T P x$ for the system $\dot{x} = Ax$ with P > 0.

Clearly $V(x) > 0$

$$\dot{V}(x) = \dot{x}^T P x + x^T P \dot{x}$$

$$\begin{aligned} &= (Ax)^T P x + x^T P (Ax) \\ &= x^T A^T P x + x^T P A x \\ &= x^T (A^T P + PA) x \end{aligned}$$

$\Rightarrow$ If $A^T P + PA < 0 (\leq 0)$, the system is asymptotically stable(Stable).

Lyapunov equation:

lyap$(A, Q) \rightarrow A^T P + PA = -Q$, if $Q > 0$, asymptotically stable; $Q \geq 0$, stable.

Calculate P

Theorem

The origin of $\dot{x} = Ax$ *is AS* $\Leftrightarrow$ *given a* $Q > 0$, $\exists$ *a unique* $P > 0$ *s.t.*

$$A^T P + PA = -Q$$

This can be easily proved by setting $P = \int_0^\infty e^{A^T t} Q e^{At} dt$

DT Lyapunov Equation

$x(k+1) = Ax(k)$

Assuming $V(k) = x^T(k)Px(k)$

$$\begin{aligned}\triangle V &= V(k+1) - V(k)\\ &= x^T(k+1)Px(k+1) - x^T(k)Px(k)\\ &= x^T A^T(k)PAx(k) - x^T(k)Px(k)\end{aligned}$$

$\Rightarrow \triangle V = x^T(k)(A^T PA - P)x(k)$

The DT Lyapunov equation is given by dlyap$(A, Q) \rightarrow \underline{A^T PA - P = -Q}$

Summary

- For Linear systems, much easier to check e-values than find P.
- Direct method is the method for non-linear systems in general
- Lyapunov equation is useful in optimal control. We will see later in this class.

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Instability

Just because **you** cannot find a lyapunov function that satisfies $\dot{V} \leq 0$ does not means instability.

Theorem

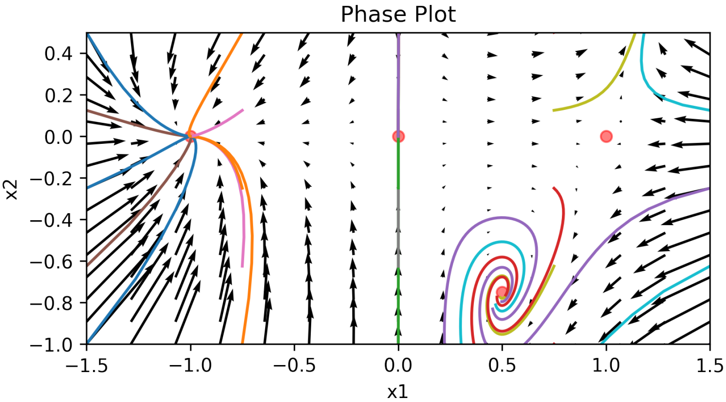
The origin of $\dot{x} = A(t)x$ *is unstable. if* $\exists V(x,t)$

1. $V(0,t) = 0, \forall t > t_0$
2. $V(x,t_0) > 0$ *for at least some point close to* $\mathbf{0}$
3. $\dot{V}(x,t) > 0$ *(Chetaev function)*

Recap: Phase Portrait Plot

$$(0,0):\ \dot{\delta}_{\mathbf{x}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \delta_{\mathbf{x}} \qquad (-1,0):\ \dot{\delta}_{\mathbf{x}} = \begin{bmatrix} -2 & -1 \\ 0 & -3 \end{bmatrix} \delta_{\mathbf{x}}$$

$$(1,0):\ \dot{\delta}_{\mathbf{x}} = \begin{bmatrix} -2 & 1 \\ 0 & 1 \end{bmatrix} \delta_{\mathbf{x}} \qquad (\tfrac{1}{2}, -\tfrac{3}{4}):\ \dot{\delta}_{\mathbf{x}} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & 0 \end{bmatrix} \delta_{\mathbf{x}}$$



Example

- Show instability of $\dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} x$.

  Try $V(x) = \frac{1}{2}x_1^2 - \frac{1}{2}x_2^2$. Consider $x(t_0) = \begin{bmatrix} \varepsilon \\ 0 \end{bmatrix} \Rightarrow V(x(t_0)) = \varepsilon^2 > 0$

  $\dot{V}(x) = x_1\dot{x}_1 - x_2\dot{x}_2 = x_1^2 + x_2^2 > 0, \forall x_1, x_2 \neq 0 \Rightarrow x = 0$ is unstable.

Example

- Show instability of $\dot{x} = \begin{bmatrix} -2 & 1 \\ 0 & 1 \end{bmatrix} x$.

  A little bit hard to find a proper V by observation. Try Lyapunov Function. To prove instability, we set $A^T P + PA = Q$. Let $V = x^T P x$. $\dot{V} = x^T (A^T P + PA) x = x^T Q x$. If Q positive definite and we can find $V > 0$ in some neighborhood of the origin, then the system is unstable.

$$\begin{bmatrix} -2 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} + \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$-2a - 2a = 4, -2b + a + b = 0, a + b - 2b = 0, b + c + b + c = 4 \Rightarrow a = b = -1, c = 3$ Let

$V(x) = x^T \begin{bmatrix} -1 & -1 \\ -1 & 3 \end{bmatrix} x = (-x_1 - x_2)x_1 + (-x_1 + 3x_2)x_2 = -x_1^2 - 2x_1x_2 + 3x_2^2$.

Consider $x(t_0) = \begin{bmatrix} 0 \\ \varepsilon \end{bmatrix} \Rightarrow V(x(t_0)) = 3\varepsilon^2 > 0$

$\dot{V}(x) = -2x_1\dot{x}_1 - 2\dot{x}_1x_2 - 2x_1\dot{x}_2 + 6x_2\dot{x}_2 = -2x_1(-2x_1 + x_2) - 2(-2x_1 + x_2)x_2$

$-2x_1x_2 + 6x_2^2 = 4x_1^2 + 4x_2^2 > 0, \forall x_1, x_2 \neq 0 \Rightarrow x = 0$ is unstable.

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BIBO & BIBS Stability

An alternative definition of stability that takes into account the forced response
Consider the LTV system

$$\begin{cases} \dot{x} = A(t)x + B(t)u \\ y = C(t)x + D(t)u \end{cases}$$

$||A(t)|| \leq M$, $||B(t)|| \leq N$, $||C(t)|| \leq O$, $||D(t)|| \leq P$

BIBO Stability: An LTV system is BIBO stable if for any $u(t)$, $||u(t)|| \leq M$, & for $x(t_0) = 0, \exists N(M, t_0) < \infty$ s.t. $||y(t)|| \leq N, \forall t \geq t_0$.

BIBS Stability: An LTV system is BIBS stable if for any $u(t)$, $||u(t)|| \leq M$, & for $x(t_0) = 0, \exists N(M, t_0) < \infty$ s.t. $||x(t)|| \leq N, \forall t \geq t_0$.

Note:This is NOT the same as Lyapunov stability. A system could be BIBO stable even if not Lyapunov stable!

Theorem

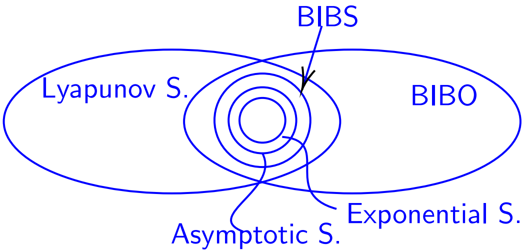
- Testing BIBO stability can be conveniently using transfer functions in the frequency domain

Theorem

Let $G_C(s) = C(sI - A)^{-1}B + D$.
A CT LTI system is BIBO stable $\Leftrightarrow$ *every pole of every* $G_{C_{ij}}$ *have negative real part.*
Let $G_D(s) = C(zI - A)^{-1}B + D$.
A DT LTI system is BIBO stable $\Leftrightarrow$ *every pole of every* $G_{D_{ij}}$ *is inside the unit circle.*

Relationships among Stability Types

- Lastly, let's consider the relationships among stability types



Example: BIBO Stable even if not Lyapunov Stable

$$\dot{x} = \begin{bmatrix} -2 & 5 \\ 0 & 3 \end{bmatrix} x + \begin{bmatrix} 4 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 7 & 8 \end{bmatrix} x + 1.5u$$

- Stability i.s.L: $\lambda_1 = -2, \lambda_2 = 3 \Rightarrow$ unstable
- BIBO Stability:

$$G(s) = \frac{(s-3)(s+20.67)}{(s-3)(s+2)} = \frac{s+20.67}{s+2}$$
$$\Rightarrow \text{ BIBO Stable!.}$$

Note: The minimal realization/Kalman Decomposition cancelled out the unstable poles with zeros.