Module 1-2: Solving Linear Dynamics

Linear Control Systems (2020)

Ding Zhao

Assistant Professor

College of Engineering

School of Computer Science

Carnegie Mellon University

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Recap: State Space Equations

	general (nonlinear)	linear
time-varying	$\dot{x} = f(x, u, t)$ $y = g(x, u, t)$	$\dot{x} = A(t)x + B(t)u$ $y = C(t)x + D(t)u$
time-invariant	$\dot{x} = f(x, u)$ $y = g(x, u)$	$\dot{x} = Ax + Bu$ $y = Cx + Du$

where $u \in \mathbb{R}^m$ is input, $x \in \mathbb{R}^n$ is the states, and $y \in \mathbb{R}^p$ is the output. In this course, we will focus on the linear SS problems.

From this lecture, I will not deliberately distinguish between scalars and vectors using the bold font, so you may see x instead of $\mathbf{x}$ that is actually vector.

Dynamics (Solutions) of Linear Time Invariant State Equations

The state space equation:

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

Properties of Matrix Exponential

Define $e^{At} = I + At + \frac{A^2t^2}{2!} + \cdots$

1. $e^{0_{n\times n}} = I$
2. $e^{A(t+\tau)} = e^{At} \cdot e^{A\tau}$
3. $e^{(A+B)t} = e^{At} \cdot e^{Bt} \Leftrightarrow AB = BA$ Proved by definition
4. $[e^{At}]^{-1} = e^{-At}$ proved by 3: $e^{At} \cdot e^{-At} = e^{0_{n\times n}} = I$
5. $e^{(A^T)} = (e^A)^T$ proved by definition
6. $\det(e^A) = e^{\mathrm{tr}(A)}$ Very useful in developing Kalman Filter
7. $\frac{d}{dt}e^{At} = Ae^{At} = e^{At}A$ proved by definition, will be used today

Dynamics (Solutions) of Linear Time Invariant State Equations

The state space equation:

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

Rewrite as $\dot{x} - Ax = Bu \Rightarrow e^{-At}\dot{x} - e^{-At}Ax = e^{-At}Bu$

Consider $\frac{d}{dt}(e^{-At}x) = -Ae^{-At}x + e^{-At}\dot{x} = e^{-At}\dot{x} - e^{-At}Ax$

Therefore $\frac{d}{dt}(e^{-At}x) = e^{-At}Bu$

$e^{-At}x|_{t_0}^{t} = \int_{t_0}^{t} e^{-A\tau}Bu(\tau)d\tau$

$e^{-At}x(t) - e^{-At_0}x(t_0) = \int_{t_0}^{t} e^{-A\tau}Bu(\tau)d\tau$

$x(t) = e^{At}e^{-At_0}x(t_0) + e^{At}\int_{t_0}^{t} e^{-A\tau}Bu(\tau)d\tau$

Dynamics (Solutions) of Linear Time Invariant State Equation

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau$$

$$y(t) = Ce^{A(t-t_0)}x(t_0) + C\int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$$

Ways to compute e^{At}

1. Apply the series definition:
   $e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}$
2. Apply Cayley-Hamilton theorem (finite polynomial - today):
   $e^{At} = \beta_0 I + \beta_1 A + \cdots + \beta_{n-1} A^{n-1}$
3. Use similarity transformations (matrix manipulation - next lecture):
   $e^{At} = M e^{Jt} M^{-1}$

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Eigenvalues and Eigenvectors

Definition

Consider a square matrix $\mathbf{A}$

An **eigenvector** for $\mathbf{A}$ is a non-null vector $\mathbf{v} \neq \mathbf{0}$ for which there exists an **eigenvalue** $\lambda \in \mathbb{R}$ such that

$$\mathbf{A}\mathbf{v} = \lambda \mathbf{v}$$

Some basic properties:

- An eigenvector has at most one eigenvalue
- If $\mathbf{v}$ is an eigenvector, then so is $a\mathbf{v}$, $\forall$ scalar $a \neq 0$
- a normalized e-vector is defined as $\mathbf{v} = \frac{\mathbf{v}}{||\mathbf{v}||_2}$

Determinant of Matrix

Minors An $n \times n$ matrix $\mathbf{A}$ contains n^2 elements a_{ij} . Each of these has associated with it a unique scalar, called a minor M_{ij} . The minor M_{pq} is the determinant of the $n-1 \times n-1$ matrix formed from $\mathbf{A}$ by crossing out the p th row and q th column.

Cofactors Each element a_{pq} of $\mathbf{A}$ has a cofactor C_{pq}, which differs from M_{pq} at most by a sign change. Cofactors are sometimes called signed minors for this reason and are given by $C_{pq} = (-1)^{p+q} M_{pq}$.

Determinants by Laplace Expansion If $\mathbf{A}$ is an $n \times n$ matrix, any arbitrary row k can be selected and $|\mathbf{A}|$ is then given by $|\mathbf{A}| = \sum_{j=1}^{n} a_{kj} C_{kj}$. Similarly, Laplace expansion can be carried out with respect to any arbitrary column l, to obtain $|\mathbf{A}| = \sum_{i=1}^{n} a_{il} C_{il}$. Laplace expansion reduces the evaluation of an $n \times n$ determinant down to the evaluation of a string of $(n-1) \times (n-1)$ determinants, namely, the cofactors.

Determinant of Matrix

Laplace, 1749-1827

Example: Calculate determinant of $\mathbf{A} = \begin{bmatrix} 2 & 4 & 1 \\ 3 & 0 & 2 \\ 2 & 0 & 3 \end{bmatrix}$.

Three of its minors are

$$\left| \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right| = ad - bc, M_{12} = \begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix} = 5, \quad M_{22} = \begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} = 4, and \quad M_{32} = \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} = 1$$

The associated cofactors are

$$C_{12} = (-1)^3 5 = -5, \quad C_{22} = (-1)^4 4 = 4, \quad C_{32} = (-1)^5 1 = -1$$

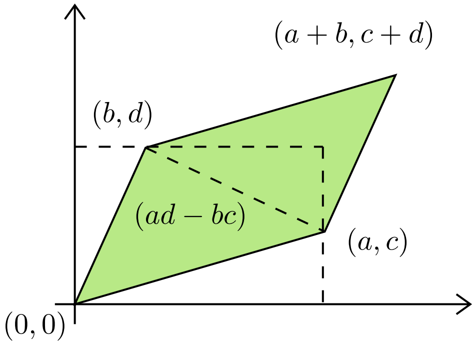
Using Laplace expansion with respect to column 2 gives $|\mathbf{A}| = 4C_{12} = -20$

Geometric Meaning of the Determinant (Group Discussion)

The area of the parallelogram is the absolute value of the determinant of the matrix formed by the vectors representing the parallelogram's sides.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

Note: Determinant is not the real volume as it can be negative.



Properties of Determinant

Given $\mathbf{A}, \mathbf{B} \in \mathbb{C}^{(n \times n)}$: $\det(\mathbf{A}) \neq 0 \iff \mathbf{A}$ is nonsingular/not defective $\iff$ rows and columns independent

Properties:

- $\det(\alpha \mathbf{A}) = \alpha^n \det(\mathbf{A})$
- $\det(\mathbf{A}^T) = \det(\mathbf{A})$
- $\det(\mathbf{I}) = 1$
- $\det(\mathbf{AB}) = \det(\mathbf{A}) \det(\mathbf{B})$
- $\det(\mathbf{A}^{-1}) = 1/\det(\mathbf{A})$
- If $\mathbf{A}$ is triangular matrix, $\det \mathbf{A} = \prod \text{diag}(\mathbf{A})$ (What if it is a diagonal matrix?)

Linear Combination

A linear combination is any **finite** sum of the form

$$\alpha_1 \mathbf{x}_1 + \alpha_2 \mathbf{x}_2 + \ldots + \alpha_n \mathbf{x}_n$$

where $\alpha_i \in \mathscr{F}, \mathbf{x}_i \in \mathscr{X}, 1 \leq i \leq n, n$ is an arbitrary integer ≥ 1

Linear Independence

- A finite set of vectors $\mathbf{x}_1, ..., \mathbf{x}_k \in \mathscr{X}$ is linearly dependent if there exist scalars $\alpha_1, ..., \alpha_k \in \mathscr{F}$ NOT ALL ZERO, such that

$$\alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2 + \ldots + \alpha_k\mathbf{x}_k = 0$$

  Otherwise, the set is linearly independent.
- The maximal number of elements in any linearly independent set of vectors in $(\mathscr{X}, \mathscr{F})$, is called the dimension of $(\mathscr{X}, \mathscr{F})$.

Question: What is the dimension of our living space.

Question: if a set $\mathscr{X}_1$ is linearly independent and $\mathscr{X}_2$ is linearly dependent, then is $\{\mathscr{X}_1, \mathscr{X}_2\}$ linearly independent? NO

Span

Let $\mathscr{S} \subset \mathscr{X}$ be a subset of $(\mathscr{X}, \mathscr{F})$.The span of $\mathscr{S}$ is the set of all linear combinations of elements of $\mathscr{S}$:

$$\text{span}\{\mathscr{S}\} = \{\mathbf{x} \in \mathscr{X} | \exists k < \infty, \mathbf{x}^1, \ldots, \mathbf{x}^k \in \mathscr{S}, \alpha_1, \ldots, \alpha_k \in \mathscr{F}, \\ \mathbf{x} = \alpha_1 \mathbf{x}^1 + \alpha_2 \mathbf{x}^2 + \ldots + \alpha_k \mathbf{x}^k\}$$

Example: Span of $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right\}$? What is the dimension?

- $(\mathbb{R}^2, \mathbb{R})$ Dim: 2

Basis

A set of vectors $\mathbf{B}$ in $(\mathscr{X}, \mathscr{F})$ is a basis iff:

1. $\mathbf{B}$ is linearly independent
2. $\text{span}\{\mathbf{B}\} = \mathscr{X}$

Characteristic Polynomial

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = 0, \mathbf{v} \neq 0 \Rightarrow (\mathbf{A} - \lambda\mathbf{I}) \text{ singular/defective} \Rightarrow \det(\lambda\mathbf{I} - \mathbf{A}) = |\lambda\mathbf{I} - \mathbf{A}| = 0$$

Define **characteristic polynomial**:

$$\Delta(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I}) = (-\lambda)^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0$$

In factored form,

$$\Delta(\lambda) = (-1)^n(\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n) = 0,$$

where the roots are $\lambda_1, \lambda_2, ..., \lambda_n$.

Note: sometimes, we may also use $\Delta(\lambda) = \det(\lambda\mathbf{I} - \mathbf{A}) = (\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n)$

Example: Eigenvalues of 3×3 Matrix

Consider the matrix

$$\begin{pmatrix} -2 & -4 & 2 \\ -2 & 1 & 2 \\ 4 & 2 & 5 \end{pmatrix}$$

The characteristic equation is

$$\det \begin{pmatrix} -2-\lambda & -4 & 2 \\ -2 & 1-\lambda & 2 \\ 4 & 2 & 5-\lambda \end{pmatrix} = 0$$

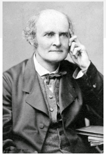
$$(-2-\lambda)[(1-\lambda)(5-\lambda) - 2 \times 2] + 4[(-2) \times (5-\lambda) - 4 \times 2] + 2[(-2) \times 2 - 4(1-\lambda)] = 0$$

$$-\lambda^3 + 4\lambda^2 + 27\lambda - 90 = 0$$

$$\lambda^3 - 4\lambda^2 - 27\lambda + 90 = (\lambda - 3)\left(\lambda^2 - \lambda - 30\right) = (\lambda - 3)(\lambda + 5)(\lambda - 6) = 0$$

Therefore, the eigenvalues are 3, -5 and 6.

Cayley Hamilton Theorem




Cayley 1821-1895
Hamilton 1805-1865

Given $\mathbf{A} \in \mathbb{R}^{n \times n}$ with characteristic polynomial $\Delta(\lambda) = \det(\lambda I - \mathbf{A}) = \lambda^n + \alpha_{n-1}\lambda^{n-1} + \ldots + \alpha_1\lambda + \alpha_0$. Then,

$$\Delta(\mathbf{A}) = \mathbf{A}^n + \alpha_{n-1}\mathbf{A}^{n-1} + \ldots + \alpha_1\mathbf{A} + \alpha_0\mathbf{I} = \mathbf{0}$$

That is, $\mathbf{A}$ satisfies its own characteristic equation.

$$\mathbf{A}^n = -\alpha_{n-1}\mathbf{A}^{n-1} - \alpha_{n-2}\mathbf{A}^{n-2} - \ldots - \alpha_1\mathbf{A} - \alpha_0\mathbf{I}$$

Cayley proved 2 by 2. Hamilton generalized the Theorem. We will learn his other theorem in the optimization section. Heuristic proof (2by2): timesλv to the right of $\mathbf{A}^2 + \alpha_1\mathbf{A} + \lambda_0 = 0 \rightarrow \lambda^2\mathbf{A}v + \alpha_1\lambda\mathbf{A}v + \alpha_0\mathbf{A}v = (\lambda^2 + \alpha_1\lambda + \alpha_0)\mathbf{A}v = 0$

Application of Cayley Hamilton Theorem: $\mathbf{A}^{-1}$

$$\Delta(\mathbf{A}) = \mathbf{A}^n + \alpha_{n-1}\mathbf{A}^{n-1} + \ldots + \alpha_1\mathbf{A} + \alpha_0\mathbf{I} = \mathbf{0}$$

A quick application:

$$-\alpha_0\mathbf{I} = \mathbf{A}(\mathbf{A}^{n-1} + \alpha_{n-1}\mathbf{A}^{n-2} + \ldots + \alpha_1)$$

If $\alpha_0 \neq 0$, we can compute

$$\mathbf{A}^{-1} = -\frac{1}{\alpha_0}(\mathbf{A}^{n-1} + \alpha_{n-1}\mathbf{A}^{n-2} + \ldots + \alpha_1)$$

Application of Cayley Hamilton Theorem: $p(\mathbf{A})$

Another application is to calculate the polynomial functions of $\mathbf{A} \in \mathbb{R}^{n\times n}$. Let $p(\mathbf{A}) = k_m\mathbf{A}^m + ... + k_1\mathbf{A} + k_0\mathbf{I}$ and m be an arbitrary integer.

$$\Delta(\mathbf{A}) = \mathbf{A}^n + \alpha_{n-1}\mathbf{A}^{n-1} + \ldots + \alpha_1\mathbf{A} + \alpha_0 I = \mathbf{0}$$

$$\Rightarrow \mathbf{A}^n = -\alpha_{n-1}\mathbf{A}^{n-1} - \alpha_{n-2}\mathbf{A}^{n-2} - \ldots - \alpha_1\mathbf{A} - \alpha_0 I$$

$$\begin{aligned}\mathbf{A}^{n+1} = \mathbf{A}\mathbf{A}^n &= -\alpha_{n-1}\mathbf{A}^n - \alpha_{n-2}\mathbf{A}^{n-1} - \ldots - \alpha_1\mathbf{A}^2 - \alpha_0\mathbf{A} \\ &= -\alpha_{n-1}(-\alpha_{n-1}\mathbf{A}^{n-1} - \ldots - \alpha_1\mathbf{A} - \alpha_0\mathbf{I}) \\ &\quad -\alpha_{n-2}\mathbf{A}^{n-1} - \ldots - \alpha_1\mathbf{A}^2 - \alpha_0\mathbf{A}\end{aligned}$$

This implies that any polynomial $p(\lambda)$, no matter its degree, can be written as

$$p(\mathbf{A}) = \beta_{n-1}\mathbf{A}^{n-1} + \ldots + \beta_1\mathbf{A} + \beta_0 I$$

Now the problem is how to get β_i

Application of Cayley Hamilton Theorem: $p(\mathbf{A})$

We can do the same thing for the characteristic equation:

$$\Delta(\lambda) = \lambda^n + \alpha_{n-1}\lambda^{n-1} + \ldots + \alpha_1\lambda + \alpha_0 = 0$$
$$\Rightarrow \lambda^n = -\alpha_{n-1}\lambda^{n-1} - \ldots - \alpha_1\lambda - \alpha_0$$

$$\begin{aligned}\lambda^{n+1} = \lambda\lambda^n &= -\alpha_{n-1}\lambda^n - \alpha_{n-2}\lambda^{n-1} - \ldots - \alpha_1\lambda^2 - \alpha_0\lambda \\ &= -\alpha_{n-1}(-\alpha_{n-1}\lambda^{n-1} - \ldots - \alpha_1\lambda - \alpha_0) \\ &\quad - \alpha_{n-2}\lambda^{n-1} - \ldots - \alpha_1\lambda^2 - \alpha_0\lambda\end{aligned}$$

$$\Rightarrow p(\lambda) = g(\lambda) = \beta_{n-1}\lambda^{n-1} + \ldots + \beta_1\lambda + \beta_0$$

where $p(\lambda)$ is an arbitrary dimension polynomial (now it is just a regular function, we know how to compute it!) and $g(\lambda)$ is an $(n-1)^{st}$ order polynomial in λ.

Extend Cayley Hamilton Theorem to Any Analytic Function

Since we know how to compute $p(\mathbf{A})$, we can actually compute any function $f(x)$ such that it can be approximated by a polynomial series that converges. Typical examples include:

- $\sin(\mathbf{A}) = \mathbf{A} - \frac{\mathbf{A}^3}{3!} + \frac{\mathbf{A}^5}{5!} - \frac{\mathbf{A}^7}{7!} + \dots$
- $\cos(\mathbf{A}) = \mathbf{I} - \frac{\mathbf{A}^2}{2!} + \frac{\mathbf{A}^4}{4!} - \frac{\mathbf{A}^6}{6!} + \dots$
- $e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2t^2}{2!} + \frac{\mathbf{A}^3t^3}{3!} + \dots \Rightarrow$

Application of Cayley Hamilton Theorem

When eigenvalues λ_i are distinct (different from each other), solve β_i from the n equations:

$$f(\lambda_1) = g(\beta_{1:n}, \lambda_1)$$

$$f(\lambda_2) = g(\beta_{1:n}, \lambda_2)$$

$$\vdots$$

$$f(\lambda_n) = g(\beta_{1:n}, \lambda_n)$$

Then calculate $f(\mathbf{A}) = g(\mathbf{A})$ given the β_i.

Example

Find matrix for e^{At} where $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$

The characteristic polynomial is $\Delta(\lambda) = (\lambda - 1)(\lambda - 3)$. The eigenvalues are $\lambda = 1, 3$. Recall: $f(\lambda) = g(\beta, \lambda) = \beta_{n-1}\lambda^{n-1} + \ldots + \beta_1\lambda + \beta_0$

$$\begin{matrix} f(\lambda) = e^{\lambda_1 t} = \beta_1\lambda_1 + \beta_0 \\ f(\lambda) = e^{\lambda_2 t} = \beta_1\lambda_2 + \beta_0 \end{matrix} \Rightarrow \begin{matrix} e^{t} = \beta_1 + \beta_0 \\ e^{3t} = 3\beta_1 + \beta_0 \end{matrix} \Rightarrow \begin{cases} \beta_0 = \frac{3e^{t} - e^{3t}}{2} \\ \beta_1 = \frac{e^{3t} - e^{t}}{2} \end{cases}$$

$$f(\mathbf{A}) = e^{\mathbf{A}t} = \beta_1\mathbf{A} + \beta_0 I = \begin{pmatrix} e^{t} & e^{3t} - e^{t} \\ 0 & e^{3t} \end{pmatrix}$$

Repeated Eigenvalues

When λ_i repeats m times, use the derivatives of f and g to populate the equations

$$\frac{df(\lambda_i)}{d\lambda_i} = \frac{dg(\lambda_i)}{d\lambda_i}$$

$$\frac{d^2 f(\lambda_i)}{d\lambda_i^2} = \frac{d^2 g(\lambda_i)}{d\lambda_i^2}$$

$$\vdots$$

$$\frac{d^{(m-1)} f(\lambda_i)}{d\lambda_i^{m-1}} = \frac{d^{(m-1)} g(\lambda_i)}{d\lambda_i^{m-1}}$$

Example: Repeated Eigenvalues

Find matrix for e^{At} where $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

Repeated eigenavlues. The characteristic polynomial is $\Delta(\lambda) = (\lambda - 1)^2$. The eigenvalues are $\lambda = 1$. Recall: $f(\lambda) = \beta_{n-1}\lambda^{n-1} + \ldots + \beta_1\lambda + \beta_0$

$$\begin{array}{c} f(\lambda) = e^{\lambda t} = \beta_1\lambda + \beta_0 \\ \frac{df(\lambda)}{d\lambda} = \frac{dg(\lambda)}{d\lambda} \Rightarrow te^{\lambda t} = \beta_1 \end{array} \Rightarrow \begin{array}{c} e^t = \beta_1 + \beta_0 \\ te^t = \beta_1 \end{array} \Rightarrow \begin{cases} \beta_0 = e^t - te^t \\ \beta_1 = te^t \end{cases}$$

$$f(\mathbf{A}) = e^{\mathbf{A}t} = \beta_1\mathbf{A} + \beta_0 I = \begin{pmatrix} e^t & 2te^t \\ 0 & e^t \end{pmatrix}$$

Why does Taking Gradient Work?

Consider calculating $\mathbf{A}^3$ for $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ using Cayley Hamilton theorem

The characteristic polynomial is $\Delta(\lambda) = \lambda^2 - 3\lambda + 2$. The eigenvalues are $\lambda = 1, 2$. Recall:
$f(\lambda) = g(\beta, \lambda) = \beta_{n-1}\lambda^{n-1} + \ldots + \beta_1\lambda + \beta_0 = \beta_1\lambda + \beta_0$

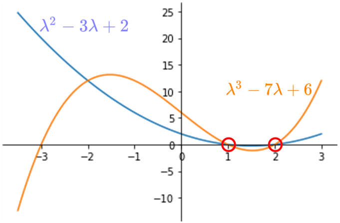
$$\begin{matrix} f(\lambda) = \lambda_1^3 = \beta_1\lambda_1 + \beta_0 \\ f(\lambda) = \lambda_2^3 = \beta_1\lambda_2 + \beta_0 \end{matrix} \Rightarrow \begin{matrix} 1 = \beta_1 + \beta_0 \\ 8 = 2\beta_1 + \beta_0 \end{matrix} \Rightarrow \begin{cases} \beta_0 = -6 \\ \beta_1 = 7 \end{cases}$$

Given $\Delta(\lambda) = 0$ and $f(\lambda) - g(\beta, \lambda) = 0$ share the same roots, the latter expression can be written as

$$f(\lambda) - g(\beta, \lambda) = \Delta(\lambda)h(\lambda)$$

$$\Rightarrow \lambda^3 - 7\lambda + 6 = (\lambda^2 - 3\lambda + 2)h(\lambda) \Rightarrow h(\lambda) = \lambda + 3$$

At $\lambda = 1, f(1) - g(\beta, 1) = 0$
At $\lambda = 2, f(2) - g(\beta, 2) = 0$



Example

Consider calculating $\mathbf{A}^3$ for $\mathbf{A} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ using Cayley Hamilton theorem

Repeated eigenvalues: Characteristic polynomial $\Delta(\lambda) = \lambda^2 - 4\lambda + 4$. The eigenvalues are $\lambda = 2$. Recall: $f(\lambda) = g(\beta, \lambda) = \beta_{n-1}\lambda^{n-1} + \ldots + \beta_1\lambda + \beta_0 = \beta_1\lambda + \beta_0$

$$\begin{matrix} f(\lambda) = \lambda^3 = \beta_1\lambda + \beta_0 \\ \frac{df(\lambda)}{d\lambda} = \frac{dg(\lambda)}{d\lambda} \Rightarrow 3\lambda^2 = \beta_1 \end{matrix} \Rightarrow \begin{matrix} 8 = 2\beta_1 + \beta_0 \\ 12 = \beta_1 \end{matrix} \Rightarrow \begin{cases} \beta_0 = -16 \\ \beta_1 = 12 \end{cases}$$

Consider

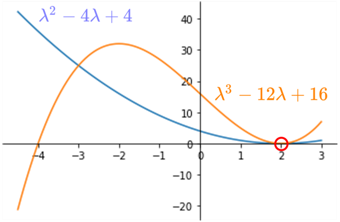
$$f(\lambda) - g(\beta, \lambda) = \Delta(\lambda)h(\lambda)$$

$$\Rightarrow \lambda^3 - 12\lambda + 16 = (\lambda^2 - 4\lambda + 4)(\lambda + 4) = 0$$

At $\lambda = 2$,

$$f(2) - g(\beta, 2) = 0$$

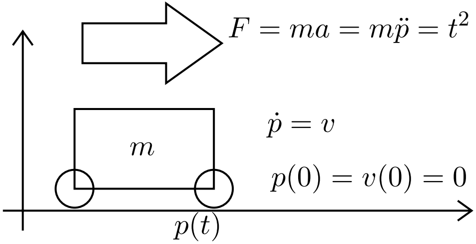
$$f'(2) - g'(\beta, 2) = 0$$



Note:

the gradient is zero at 2

Solve the Longitudinal Driving Example (Conventional Way)



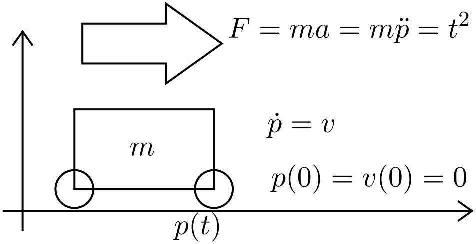
Which is the state?:
Solve $p(t)$

$$\ddot{p}(t) = a = F(t)/m$$

$$\dot{p}(t) = \dot{p}(0) + \int_0^t \ddot{p}(t) d\tau$$
$$= \int_0^t \frac{F(t)}{m} d\tau = \int_0^t \frac{t^2}{m} d\tau = \frac{t^3}{3m}$$

$$p(t) = p(0) + \int_0^t \dot{p}(t) d\tau = \int_0^t \frac{t^3}{3m} d\tau = \frac{t^4}{12m}$$

Solve the Longitudinal Driving Example (State Space) (Group Discussion)



Solve $p(t)$

$$x_1 = p \ , \ x_2 = \dot{p}, \ \mathbf{x} = [x_1, x_2]^T$$

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} F$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}$$

Solve the Longitudinal Driving Example (Cayley-Hamilton)

E-values 0 and 0. Use C-H to find e^{At}

$$\begin{cases} e^{0t} = \beta_1 \cdot 0 + \beta_0 \Rightarrow \beta_0 = 1 \\ te^{0t} = \beta_1 \Rightarrow \beta_1 = t \end{cases} \Rightarrow e^{At} = t \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + I = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

Calculate $y(t)$ with e^{At}

$$y(t) = Ce^{A(t-t_0)}x(t_0) + C\int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \int_0^t \begin{bmatrix} 1 & t-\tau \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1/m \end{bmatrix} \tau^2 d\tau = \begin{bmatrix} 1 & 0 \end{bmatrix} \int_0^t \begin{bmatrix} (t-\tau)/m \\ 1/m \end{bmatrix} \tau^2 d\tau$$

$$= \int_0^t \frac{t-\tau}{m}\tau^2 d\tau = \int_0^t \frac{t\tau^2}{m} - \frac{\tau^3}{m} d\tau = (\frac{t}{3m}\tau^3 - \frac{\tau^4}{4m})|_0^t = t^4(\frac{1}{3m} - \frac{1}{4m}) = \frac{t^4}{12m}$$

Recap: Cayley Hamilton Theorem

Given $\mathbf{A} \in \mathbb{R}^{n \times n}$ with characteristic polynomial $\Delta(\lambda) = \det(\lambda I - \mathbf{A}) = \lambda^n + \alpha_{n-1}\lambda^{n-1} + \ldots + \alpha_1\lambda + \alpha_0$. Then,

$$\Delta(\mathbf{A}) = \mathbf{A}^n + \alpha_{n-1}\mathbf{A}^{n-1} + \ldots + \alpha_1\mathbf{A} + \alpha_0\mathbf{I} = \mathbf{0}$$

That is, $\mathbf{A}$ satisfies its own characteristic equation.

$$\mathbf{A}^n = -\alpha_{n-1}\mathbf{A}^{n-1} - \alpha_{n-2}\mathbf{A}^{n-2} - \ldots - \alpha_1\mathbf{A} - \alpha_0\mathbf{I}$$

Recap: Application of the Cayley Hamilton Theorem

Given $\mathbf{A} \in \mathbb{R}^{n \times n}$ with eigenvalues λ_i, let $f(x)$ be an arbitrary scalar function and $g(x)$ an $(n-1)^{st}$ order polinomial. When eigenvalues λ_i are distinct, solve β_i from the n equations:

$$
\begin{aligned}
f(\lambda_1) &= g(\beta_1, \lambda_1) \\
f(\lambda_2) &= g(\beta_2, \lambda_2) \\
&\vdots \\
f(\lambda_n) &= g(\beta_n, \lambda_n)
\end{aligned}
$$

Then calculate $f(\mathbf{A}) = g(\mathbf{A})$ given the β_i.

When λ_i repeats m times, use the derivatives of f and g to populate the equations

$$
\begin{aligned}
\frac{df(\lambda_i)}{d\lambda_i} &= \frac{dg(\lambda_i)}{d\lambda_i} \\
\frac{d^2 f(\lambda_i)}{d\lambda_i^2} &= \frac{d^2 g(\lambda_i)}{d\lambda_i^2} \\
&\vdots \\
\frac{d^{(m-1)} f(\lambda_i)}{d\lambda_i^{m-1}} &= \frac{d^{(m-1)} g(\lambda_i)}{d\lambda_i^{m-1}}
\end{aligned}
$$

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Discrete-Time Linear Time Invariant Systems

$$
\begin{aligned}
x(1) &= Ax(0) + Bu(0) \\
x(2) &= Ax(1) + Bu(1) = A^2x(0) + ABu(0) + Bu(1) \\
&\vdots \\
x(k) &= A^k x(0) + \sum_{m=0}^{k-1} A^{k-m-1} Bu(m) \\
y(k) &= CA^k x(0) + \sum_{m=0}^{k-1} CA^{k-m-1} Bu(m) + Du(k)
\end{aligned}
$$

Need to compute $\mathbf{A}^k$

Similarity Transformation

A square matrix can always be decomposed by a **similarity transformation**, which is a relationship between two square matrices $\mathbf{A}$ and $\hat{\mathbf{A}}$ of the form

$$\hat{\mathbf{A}} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S} \Leftrightarrow \mathbf{A} = \mathbf{S}\hat{\mathbf{A}}\mathbf{S}^{-1}$$

for any nonsingular matrix $\mathbf{S}$.
Here $\hat{\mathbf{A}}$ is in either diagonal or Jordan form. In either case,

$$\mathbf{A}^k = \mathbf{S}\hat{\mathbf{A}}\mathbf{S}^{-1}\mathbf{S}\hat{\mathbf{A}}\mathbf{S}^{-1}\cdots\mathbf{S}\hat{\mathbf{A}}\mathbf{S}^{-1} = \mathbf{S}\hat{\mathbf{A}}^k\mathbf{S}^{-1}$$

is much easier to compute. We will see that actually $e^{\hat{\mathbf{A}}t}$ is easier to compute too.
Let us first recap matrix inverse.

Recap: Matrix Inverse

In order to learn similarity transformation, we need review the definition of matrix inverse.
If matrix $\mathbf{A}$ is square, and (square) matrix $\mathbf{B}$ satisfies

$$\mathbf{BA} = \mathbf{AB} = \mathbf{I}$$

then $\mathbf{B}$ is called the inverse of $\mathbf{A}$ and is denoted as $\mathbf{B} = \mathbf{A}^{-1}$ For the inverse to exist, $\mathbf{A}$ must have a nonzero determinant, i.e., $\mathbf{A}$ must be non-singular. When this is true, $\mathbf{A}$ has a unique inverse given by

$$\mathbf{A}^{-1} = \frac{\mathbf{C}^T}{|\mathbf{A}|}$$

where $\mathbf{C}$ is the matrix formed by the cofactors C_{ij}. The matrix $\mathbf{C}^T$ is called the adjoint matrix, $\mathrm{Adj}(\mathbf{A})$. Thus the inverse of a nonsingular matrix is

$$\mathbf{A}^{-1} = \mathrm{Adj}(\mathbf{A})/|\mathbf{A}|$$

Recap: Matrix Inverse Properties

- $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$, i.e., inverse of inverse is original matrix (assuming $\mathbf{A}$ is **invertible**)
- $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$ (assuming $\mathbf{A}, \mathbf{B}$ are invertible)
- $(\mathbf{A}^T)^{-1} = (\mathbf{A}^{-1})^T$ (assuming $\mathbf{A}$ is invertible)
- $\mathbf{I}^{-1} = \mathbf{I}$
- $(\alpha\mathbf{A})^{-1} = (1/\alpha)\mathbf{A}^{-1}$ (assuming $\mathbf{A}$ invertible, $\alpha \neq 0$)
- $\begin{pmatrix} \mathbf{A}_1 & 0 & \cdots & 0 \\ 0 & \mathbf{A}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{A}_n \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{A}_1^{-1} & 0 & \cdots & 0 \\ 0 & \mathbf{A}_2^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{A}_n^{-1} \end{pmatrix}$
- $\mathbf{A}^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

How to compute the inverse of high dimensional matrices

Recap: Elementary Operation

How to compute the inverse? We can use Gaussian Elimination. First need to know the three basic operations on a matrix, called elementary operations:

1. Row switching: The interchange of two rows (or of two columns).
2. Row multiplication: The multiplication of every element in a given row (or column) by a scalar α.
3. Row addition: The multiplication of the elements of a given row (or column) by a scalar α, and adding the result to another row (column). The original row (column) is unaltered. We will mainly use row operation in this course.

Recap: Gaussian Elimination

Gauss, 1777-1855

Highlights of Gaussian Elimination:

Use elementary row operations to reduce the augmented matrix to a form such that

1. The first non zero entry of each row should be on the right-hand side of the first non zero entry of the preceding row. Simply put, the coefficient part (corresponding to $\mathbf{A}$) of the augmented matrix should form an $n \times n$ upper triangular matrix.
2. Any zero row should be at the bottom of the matrix.

Gaussian elimination uses the row reduced form of the augmented matrix to compactly solve a given system of linear equations (echelon form).

Example

$$\mathbf{Ax} = \mathbf{y}; \mathbf{A} = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}; \ \mathbf{y} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$$

$$[\mathbf{A}|\mathbf{y}] = \begin{bmatrix} 2 & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & 9 \end{bmatrix}$$

Subtract twice of first row from the second and add first equation to the third row.

$$= \begin{bmatrix} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ 0 & 8 & 3 & 14 \end{bmatrix}$$

Example cont.

Add row 2 to the third row.

$$= \begin{bmatrix} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

The diagonal elements: 2,-8 and 1 are the pivot elements.
Rank of $\mathbf{A}$ is defined as the number of non-zero rows of the first n columns.

Use Gaussian Elimination for Matrix Inverse (Group Discussion)

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \Rightarrow [A|I] = \left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\textbf{Gaussian Elimination} \Rightarrow [I|B] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{array}\right].$$

How do Computers do Gaussian Elimination - LU decomposition

Consider a system of 3 equations. For $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, we have $\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$ and $\mathbf{U} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$ such that $\mathbf{A} = \mathbf{LU}$. The upper triangular matrix $\mathbf{U}$ formed is equivalent to the result of applying Gaussian elimination on $\mathbf{A}$

$$
\begin{aligned}
\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \\
&= \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix}
\end{aligned}
$$

We use this to find the entries in $\mathbf{L}$ and $\mathbf{U}$.

Example: LU decomposition for Gaussian Elimination

$$\mathbf{A} = \begin{bmatrix} 2. & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}; \; \mathbf{y} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$$

Considering LU decomposition, we have $\mathbf{A} = \mathbf{LU}$ which gives

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$$

$$u_{11} = 2 \qquad u_{12} = 1 \qquad u_{13} = 1$$

$$l_{21}u_{11} = 4 \quad \Rightarrow l_{21} \times 2 = 4 \quad \Rightarrow \quad l_{21} = 2$$

$$l_{21}u_{12} + u_{22} = -6 \quad \Rightarrow 2 \times 1 + u_{22} = -6 \quad \Rightarrow \quad u_{22} = -8$$

$$l_{21}u_{13} + u_{23} = 0 \quad \Rightarrow 2 \times 1 + u_{23} = 0 \quad \Rightarrow \quad u_{23} = -2$$

Example cont.

Similarly, we find $l_{31} = -1, l_{32} = -1, u_{33} = 1$ Therefore, we obtain

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}, \quad \mathbf{U} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

The next step is to solve $\mathbf{Lz} = \mathbf{y}$ for the vector $\mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \mathbf{Ux}$ i.e. we consider

$$\mathbf{Lz} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix} = \mathbf{y} \Rightarrow \begin{array}{c} z_1 = 5 \\ 2z_1 + z_2 = -2 \\ -z_1 - z_2 + z_3 = 9 \end{array} \Rightarrow \begin{array}{c} z_1 = 5 \\ z_2 = -12 \\ z_3 = 2 \end{array}$$

Example cont.

Now that we have found $\mathbf{z}$ we finish the procedure by solving $\mathbf{Ux} = \mathbf{z}$ for $\mathbf{x}$. That is we solve

$$\mathbf{Ux} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -12 \\ 2 \end{bmatrix} = \mathbf{z} \Rightarrow \begin{matrix} 2x_1 + x_2 + x_3 = 5 \\ -8x_2 - 2x_3 = -12 \\ x_3 = 2 \end{matrix} \Rightarrow \begin{matrix} x_1 = 1 \\ x_2 = 1 \\ x_3 = 2 \end{matrix}$$

Therefore we have found that the solution to the given system of simultaneous equations is

$$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

Another Way to Check Invertibility Rank

$(\mathscr{X}, \mathscr{F})$ and $(\mathscr{Y}, \mathscr{F})$ are vector spaces. $\mathscr{A} : \mathscr{X} \rightarrow \mathscr{Y}$ is a linear transformation.

$$\mathbf{y} = \mathbf{A}\mathbf{x}$$

- Null space:

  $$\mathscr{N}(\mathbf{A}) = \{\mathbf{x} \in \mathscr{X} | \mathbf{A}\mathbf{x} = \mathbf{0}\}$$

  Nullity of $\mathbf{A} = dim \mathscr{N}(\mathbf{A})$
- Range space (linear combination of columns):

  $$\mathscr{R}(\mathbf{A}) = \{\mathbf{y} \in \mathscr{Y} | \exists \mathbf{x} \in \mathscr{X}, \text{ such that } \mathbf{y} = \mathbf{A}\mathbf{x}\}$$

  Rank of $\mathbf{A} = dim \mathscr{R}(\mathbf{A})$

Example of Range Space and Null Space

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

- $\mathscr{N}(\mathbf{A}) =$ span $\{[1,0,0]^T\} \Rightarrow$ Nullity of $\mathbf{A} = 1$
- $\mathscr{R}(\mathbf{A}) =$ span $\{[1,0,0]^T, [0,1,0]^T\} \Rightarrow \mathrm{Rank}$ of $\mathbf{A} = 2$

Note: r($\mathbf{A}$)=number of pivot elements in Gaussian elimination. Any column without a pivot is a free variable. Note: $\mathbf{A}$ needs to have full rank to be invertable.

Summary of Ways to Check Invertibility

Given $\mathbf{A} \in \mathbb{R}^{n \times n}$

- $\mathbf{A}$ has full rank; that is, $r(\mathbf{A}) = n$
- $\det(\mathbf{A}) \neq 0$
- All columns (or rows) of $\mathbf{A}$ are linear independent
- The null space of $\mathscr{N}(\mathbf{A}) = \{\mathbf{0}\}$

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for any nonsingular matrix $\mathbf{S}$.
Here $\hat{A}$ is in either diagonal or Jordan form. In either case,

$$\mathbf{A}^k = \mathbf{S}\hat{\mathbf{A}}\mathbf{S}^{-1}\mathbf{S}\hat{\mathbf{A}}\mathbf{S}^{-1}\cdots\mathbf{S}\hat{\mathbf{A}}\mathbf{S}^{-1} = \mathbf{S}\hat{\mathbf{A}}^k\mathbf{S}^{-1}$$

is much easier to compute. We will see that actually $e^{\hat{A}t}$ is easier to compute too.

Diagonalizability (Distinct Eigenvalues)

Let $\mathbf{A}$ have distinct eigenvalues $\lambda_i, 1 \leq i \leq n$ with corresponding eigenvectors $\{\mathbf{v}_1 \ldots \mathbf{v}_n\}$.

$$\mathbf{A}\mathbf{v}_i = \lambda_i \mathbf{v}_i$$

$$\mathbf{A}\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \ldots & \mathbf{v}_n\end{bmatrix} = \begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \ldots & \mathbf{v}_n\end{bmatrix}\begin{bmatrix}\lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n\end{bmatrix}$$

Define the **modal matrix** $\mathbf{M} = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n]$. Because $\mathbf{v}_1 \ldots \mathbf{v}_n$ are independent, $\mathbf{M}$ is nonsingular. $\mathbf{A}$ can be diagonalized as

$$\mathbf{A}\mathbf{M} = \mathbf{M}\mathbf{\Lambda} \Rightarrow \mathbf{A} = \mathbf{M}\begin{bmatrix}\lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n\end{bmatrix}\mathbf{M}^{-1} = \mathbf{M}\mathbf{\Lambda}\mathbf{M}^{-1}$$

Note here $M = S^{-1}$ for conventional reason.

Examples

Type I (distinct eigenvalues):

$\mathbf{A} = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1/2 \end{bmatrix} \mathbf{x}_4 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, compute $\mathbf{A}^{2020}$. Choose e-vectors from $\mathbf{x}_i$

$\lambda_1 = 3, \lambda_2 = 2, \lambda_3 = 1, \mathbf{v}_1 = \mathbf{x}_1, \mathbf{v}_2 = \mathbf{x}_4, v_3 = \mathbf{x}_3$ $\mathbf{M} = [\mathbf{x}_1, \mathbf{x}_4, \mathbf{x}_3]$ Use Gaussian elimination:

$$\mathbf{A} = [\mathbf{M}|\mathbf{I}] = \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1/2 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 \end{array}\right] \Rightarrow \mathbf{M}^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\mathbf{A} = \mathbf{M}\boldsymbol{\Lambda}\mathbf{M}^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\mathbf{A}^{2020} = \mathbf{M}\boldsymbol{\Lambda}^{2020}\mathbf{M}^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 3^{2020} & 0 & 0 \\ 0 & 2^{2020} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Characteristic Polynomial with Repeated Roots

We encounted this problem with Cayley Hamilton theorem. Here we need a different solution. If there are $p < n$ **distinct roots**,

$$\Delta(\lambda) = (\lambda - \lambda_1)^{m_1}(\lambda - \lambda_2)^{m_2} \cdots (\lambda - \lambda_p)^{m_p}$$

m_i is called the **algebraic multiplicity** of the eigenvalue λ_i, with $m_1 + m_2 + \cdots + m_p = n$. λ_i called as an m_i-order root

The dimension of this space is called the **geometric mutiplicity** of λ_i

$$\begin{aligned} q_i &= \#\text{of linearly independent eigenvector of } \lambda_i \\ &= n - \operatorname{rank}(A - \lambda_i \mathbf{I}) \leq m_i \end{aligned}$$

Examples cont.

Type II$_1$ Matrix of decomposition:

$$\mathbf{A} = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1/2 \end{bmatrix} \mathbf{x}_4 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$, Choose e-vectors from $\mathbf{x}_i$

$n = 3, p = 2$. For $\lambda_1 = 3, m_1 = 1, q_1 = 1$, we have evector $\mathbf{v}_1 = \mathbf{x}_1$. For $\lambda_2 = 2, m_1 = 2$, we have two eigenvectors $\mathbf{v}_2 = \mathbf{x}_3, \mathbf{v}_3 = \mathbf{x}_4$. So $q_2 = 2$.

$$\mathbf{A} = \mathbf{M}\mathbf{\Lambda}\mathbf{M}^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Similar to Type I.

Examples cont.

Type II_2 $(q_i = 1)$ $\mathbf{A} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix}, \mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1/2 \end{bmatrix} \mathbf{x}_4 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, Choose e-vectors from $\mathbf{x}_i$

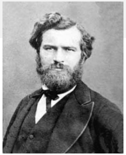
Decomposition: $n = 3, p = 1$. For $\lambda_1 = 2, m_1 = 3, \mathbf{A} - \lambda_1 \mathbf{I} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$,

$q_1 = n - rank(\mathbf{A} - \lambda_1 \mathbf{I}) = 3 - 2 = 1$

We can only find a single eigenvector $\mathbf{v}_1 = \mathbf{x}_1$ for the null space of $\mathbf{A} - \lambda_1 \mathbf{I}$. $q_1 = 1$. $\mathbf{A}$ is **defective**. The total number of independent e-vectors is smaller than n and this **sadly** means $\mathbf{A}$ can not be fully decoupled!!

So we do what we can and try to decompose $\mathbf{A}$ as much as possible. $\Rightarrow$ Jordan Decomposition. The key problem is that we do not have enough eigenvectors for the repeated λ_i. So we need to generate some "fake" but reasonable e-vectors. We call them "generated e-vectors".

Jordan Decomposition (Jordan Canonical Form)


Jordan, 1838-1922

Consider λ_1 with $p = 1$, $m_1 = n, q_1 = 1$. Then $m_1 - q_1$ generalized eigenvectors are required. Let $\mathbf{A}\mathbf{v}_1 = \lambda_1 \mathbf{v}_1$ ←The usual e-vector

$\mathbf{A}\mathbf{v}_2 = \lambda_1 \mathbf{v}_2 + \mathbf{v}_1$ ←Add the $\mathbf{v}_1$ term artificially

$\vdots$

$\mathbf{A}\mathbf{v}_{m_1} = \lambda_1 \mathbf{v}_{m_1} + \mathbf{v}_{m_1 - 1}$

Now we have

$$\mathbf{A}[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \cdots \mathbf{v}_{n-1} \quad \mathbf{v}_n] = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \cdots \mathbf{v}_{n-1} \quad \mathbf{v}_n] \begin{bmatrix} \lambda_1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_1 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \lambda_1 & \cdots & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & \cdots & \lambda_1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda_1 \end{bmatrix}$$

$\Rightarrow \mathbf{AM} = \mathbf{MJ}$, we call $\mathbf{J}$ a Jordan block.

$$\mathbf{A} = \mathbf{MJM}^{-1}$$

Examples cont.

Type II$_2$ $(q_i = 1)$ $\mathbf{A} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix}$, $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1/2 \end{bmatrix}$ $\mathbf{x}_4 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, Choose e-vectors from $\mathbf{x}_i$

Let us pick $\mathbf{x}_1 = \mathbf{v}_1$ and generalized $\mathbf{v}_2$ from $(\mathbf{A} - \lambda_1 \mathbf{I})\mathbf{v}_2 = \mathbf{v}_1$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \mathbf{x}_2$$

$$(\mathbf{A} - \lambda_1 \mathbf{I})\mathbf{v}_3 = \mathbf{v}_2 \Rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \mathbf{x}_3$$

$$\mathbf{A} = \mathbf{M}\mathbf{\Lambda}\mathbf{M}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Examples cont.

Type II$_3$ $(1 < q_i < m_i)$,$\mathbf{A} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix}$, $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1/2 \end{bmatrix}$ $\mathbf{x}_4 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, Choose e-vectors from $\mathbf{x}_i$

$p = 1, m_1 = 3, \lambda_1 = 2, \mathbf{A} - \lambda_1 \mathbf{I} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$, $q = n -$ rank$(\mathbf{A} - \lambda_1 \mathbf{I}) = 2$, we need 1 generated e-vector. Let $\mathbf{v}_1 = \mathbf{x}_1, \mathbf{v}_2 = \mathbf{x}_2$. We can not find a $\mathbf{v}_3$ satisfying $(\mathbf{A} - \lambda_1 \mathbf{I})\mathbf{v}_3 = \mathbf{v}_1$.

Let $(\mathbf{A} - \lambda_1 \mathbf{I})\mathbf{v}_3 = \mathbf{v}_2$. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Choose $\mathbf{v}_3 = \mathbf{x}_3$,

$$\mathbf{M} = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3], \mathbf{A} = \mathbf{M}\mathbf{\Lambda}\mathbf{M}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 2 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Examples cont.

Type II$_3$ $(1 < q_i < m_i)$

We can also choose $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1/2 \end{bmatrix}$ or $\begin{bmatrix} 0 \\ 1 \\ 1/2 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 1 \\ 1/2 \end{bmatrix}$. It will not change the Jordan form. Let COLAB LINK to do it quickly.

You can also let $\mathbf{v}_1 = \mathbf{x}_2$ and $\mathbf{v}_2 = \mathbf{x}_1$. In this case, $\mathbf{v}_3$ will be chained with $\mathbf{v}_1$ and place on the right side of it. If $\mathbf{M} = [\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_2]$ $J = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ to reflect the generation of $\mathbf{v}_3$ from $\mathbf{v}_1$.

Jordan normal form is sometimes called "Jordan Canonical form". But, it is actually not fully canonical. This is one of the limitations of the Jordan decomposition. We will learn some canonical decomposition later with controllable and observable canonical forms.

Jordan Decomposition In General

Let $\mathbf{A}$ be an $n \times n$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ of algebraic multiplicities $m_1, ..., m_p$ and geometric multiplicities $q_1, ..., q_p$. Then $\exists$ an invertible matrix $\mathbf{M}$ such that $\mathbf{J} = \mathbf{M}^{-1}\mathbf{AM}$, where

$$\mathbf{J} = \begin{bmatrix} \hat{\mathbf{J}}_1 & 0 & 0 & 0 \\ 0 & \hat{\mathbf{J}}_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \hat{\mathbf{J}}_p \end{bmatrix}_{n \times n}$$

$\#$blocks $= p$($\#$distinct e-values)

$$\hat{\mathbf{J}}_i = \begin{bmatrix} \hat{\mathbf{J}}_{i1} & 0 & 0 & 0 \\ 0 & \hat{\mathbf{J}}_{i2} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \hat{\mathbf{J}}_{iq_i} \end{bmatrix}_{m_i \times m_i}$$

$$\hat{\mathbf{J}}_{ij} = \begin{bmatrix} \lambda_i & 1 & 0 \\ 0 & \ddots & 1 \\ 0 & 0 & \lambda_i \end{bmatrix}_{? \times ?, ? \geq 2}$$

$\#$blocks $= q_i$($\#$indep e-vectors assoc. with λ_i)

In general, the dimensions for the 3rd level Jordan blocks are undetermined, except in type I (dim $= 1$), II$_1$ (1), or II$_2$ (m_i).

Another Example of Jordan Decomposition

$\mathbf{A}$ is 7×7. Distinct $\lambda_1, \lambda_2, m_1 = 5, m_2 = 2, q_1 = 2, q_2 = 1$
What are the possible Jordan blocks?

$$\hat{\mathbf{J}} = \begin{bmatrix} \hat{\mathbf{J}}_1 & 0 \\ 0 & \hat{\mathbf{J}}_2 \end{bmatrix}$$

$$\hat{\mathbf{J}}_2 = \begin{bmatrix} \lambda_2 & 1 \\ 0 & \lambda_2 \end{bmatrix}$$

$$\hat{\mathbf{J}_1} = \begin{bmatrix} \hat{\mathbf{J}}_{11} & 0 \\ 0 & \hat{\mathbf{J}}_{12} \end{bmatrix}$$

The dimension of $\hat{\mathbf{J}}_{11}$ and $\hat{\mathbf{J}}_{12}$ can vary based on how we want to build the chain of generated e-vectors. It can be either (1,4) or (2,3)

$$\hat{\mathbf{J}}_1 = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 1 & 0 \\ 0 & 0 & 0 & \lambda_1 & 1 \\ 0 & 0 & 0 & 0 & \lambda_1 \end{bmatrix}$$

$$\hat{\mathbf{J}}_1 = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1 & 1 & 0 \\ 0 & 0 & 0 & \lambda_1 & 1 \\ 0 & 0 & 0 & 0 & \lambda_1 \end{bmatrix}$$

- **Similarity transformations**: $\hat{\mathbf{A}} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}$
- **Model matrix** $\mathbf{M} = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n]$, where $\mathbf{v}_1 \ldots \mathbf{v}_n$ are independent eigenvectors.
- Type I, $p = n, m_i = q_i = 1$ (distinct eigenvalues): $\mathbf{\Lambda} = \mathbf{M}^{-1}\mathbf{A}\mathbf{M}$, where $\mathbf{\Lambda}$ is diagonal.
- Type II$_1$, $p < n$, $m_i = q_i > 0$, $\mathbf{\Lambda} = \mathbf{M}^{-1}\mathbf{A}\mathbf{M}$ (non-defective)
- Type II$_2$, $p = 1 < n$, $m_1 = n$, $q_1 = 1$ (a single Jordan block): $\mathbf{J} = \mathbf{M}^{-1}\mathbf{A}\mathbf{M}$, where $\mathbf{J} = \begin{bmatrix} \lambda_i & 1 & 0 \\ 0 & \ddots & 1 \\ 0 & 0 & \lambda_i \end{bmatrix}$, where $\mathbf{v}_1$ is e-vector and $\mathbf{v}_i, i > 1$ are **generated eigenvectors**:
  $\mathbf{A}\mathbf{v}_1 = \lambda_1\mathbf{v}_1, \mathbf{A}\mathbf{v}_2 = \lambda_1\mathbf{v}_2 + \mathbf{v}_1 \cdots, \mathbf{A}\mathbf{v}_{m_i} = \lambda_1\mathbf{v}_{m_i} + \mathbf{v}_{m_i-1}$
- Type II$_3$, $p < n$, $m_i > q_i > 1$ (Jordan Decomposition):

Recap: Jordan Decomposition In General

Let $\mathbf{A}$ be an $n \times n$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ of algebraic multiplicities $m_1, ..., m_p$ and geometric multiplicities $q_1, ..., q_p$. Then $\exists$ an invertible matrix $\mathbf{M}$ such that $\mathbf{J} = \mathbf{M}^{-1}\mathbf{A}\mathbf{M}$, where

$$\mathbf{J} = \begin{bmatrix} \hat{\mathbf{J}}_1 & 0 & 0 & 0 \\ 0 & \hat{\mathbf{J}}_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \hat{\mathbf{J}}_p \end{bmatrix}_{n\times n}$$

$\#$blocks $= p(\#$distinct e-values)

$$\hat{\mathbf{J}}_i = \begin{bmatrix} \hat{\mathbf{J}}_{i1} & 0 & 0 & 0 \\ 0 & \hat{\mathbf{J}}_{i2} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \hat{\mathbf{J}}_{iq_i} \end{bmatrix}_{m_i\times m_i} \qquad \hat{\mathbf{J}}_{ij} = \begin{bmatrix} \lambda_i & 1 & 0 \\ 0 & \ddots & 1 \\ 0 & 0 & \lambda_i \end{bmatrix}_{?\times ?, ?\geq 2}$$

$\#$blocks $= q_i($ $\#$indep e-vectors assoc. with $\lambda_i)$

In general, we do not know what is the dimensions for the 3rd level Jordan blocks except in type I, II_1, or II_2.

Matrix Exponential

Theorem: Suppose that A and J are similar matrices. Then, so are e^{At} and e^{Jt}. In particular, if $A = MJM^{-1}$, then $e^{At} = Me^{Jt}M^{-1}$.

Proof: $A^k = MJ^kM^{-1}$. So plug in

$e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} = \sum_{k=0}^{\infty} M \frac{J^k t^k}{k!} M^{-1} = M \sum_{k=0}^{\infty} \frac{J^k t^k}{k!} M^{-1} = Me^{Jt}M^{-1}$

Conclusion: To compute e^{At}, it is enough to know how to compute e^{Jt}, for J in Jordan Canonical form

$$J = \left[\begin{array}{cc|c} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ \hline 0 & 0 & \lambda_2 \end{array}\right] = \left[\begin{array}{c|c} J_1 & 0 \\ \hline 0 & J_2 \end{array}\right]$$

$$e^{Jt} = \left[\begin{array}{c|c} e^{J_1 t} & 0 \\ \hline 0 & e^{J_2 t} \end{array}\right] = \left[\begin{array}{cc|c} e^{\lambda_1 t} & te^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_1 t} & 0 \\ \hline 0 & 0 & e^{\lambda_2 t} \end{array}\right]$$

Exponential of Jordan Form cont.

Given $J = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, find e^{tJ}

$J = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} \Rightarrow e^{tJ} = \begin{bmatrix} e^{tJ_1} & 0 \\ 0 & e^{tJ_2} \end{bmatrix}$

$J_1 = D + N = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$e^{tD} = \begin{pmatrix} e^{2t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{2t} \end{pmatrix}$. Need to compute e^{tN}

Exponential of Jordan Form

Apply Cayley-Hamilton theorem:

$$\lambda = 0, f(N) = e^{tN}, N^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$f(\lambda) = e^{t\lambda} = 1, f'(\lambda) = te^{t\lambda} = t, f''(\lambda) = t^2 e^{t\lambda} = t^2$$

$$g(\lambda) = \beta_2\lambda^2 + \beta_1\lambda + \beta_0 = \beta_0, g'(\lambda) = 2\beta_2\lambda + \beta_1 = \beta_1, g''(\lambda) = 2\beta_2$$

$$f(\lambda) = g(\lambda), f'(\lambda) = g'(\lambda), f''(\lambda) = g''(\lambda) \Rightarrow \beta_0 = 1, \beta_1 = t, \beta_2 = \frac{1}{2}t^2$$

$$e^{tN} = f(N) = g(N) = \frac{1}{2}t^2 N^2 + tN + I = \begin{bmatrix} 1 & t & \frac{1}{2}t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$$

Finally, because $DN = ND$, $e^{tJ_1} = e^{t(D+N)} = e^{tD} \cdot e^{tN} = \begin{bmatrix} e^{2t} & te^{2t} & \frac{1}{2}t^2 e^{2t} \\ 0 & e^{2t} & te^{2t} \\ 0 & 0 & e^{2t} \end{bmatrix}$

Exponential of Jordan Form

$$e^{Jt} = \begin{bmatrix} e^{\lambda_1 t} & 0 & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & te^{\lambda_2 t} & \cdots & 0 \\ 0 & 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \end{bmatrix}$$

e^{Jt} has terms of the form $t^m e^{\lambda_i t}$, with $m \neq 0$ for Jordan blocks of order > 1. We will use this trick again in the stability analysis.

Modal Decomposition

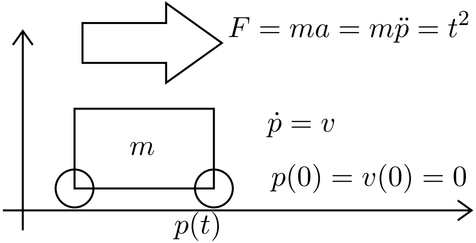
Using a similarity transformation, we can convert the state equation into diagonal (or Jordan) form.

Let $x = Mx'$ where $M = [v_1|v_2|\cdots|v_n]$ are the (generated) eigenvectors of A

$$\begin{cases} \dot{x}' = M^{-1}\hat{\mathbf{A}}Mx' + M^{-1}Bu \\ y = CMx' + Du \end{cases}$$

Here $\hat{A} = M^{-1}AM$ is in either diagonal or Jordan form. In either case, $e^{\hat{A}t}$ is easier to compute.

Revisiting the same example ...



Which is the state?:
$\{p, \dot{p}, \ddot{p}\}$, $\{\dot{p}, \ddot{p}\}$, $\{p, \dot{p}\}$, $\{p\}$
Solve $p(t)$

$$x_1 = p \ , \ x_2 = \dot{p}, \ \mathbf{x} = [x_1, x_2]^T$$

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} F$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}$$

Application

E-values 0 and 0. Use Jordan form to find e^{At}

$\lambda = 0, m = 2 \rightarrow \mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, q = n - \text{rank}(\mathbf{A} - \lambda_1\mathbf{I}) = 1 < m \Rightarrow J = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and generalized $\mathbf{v}_2$ from $(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{v}_2 = \mathbf{v}_1$ gives $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$$A = MJM^{-1} \Rightarrow e^{At} = Me^{Jt}M^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^{0t} & te^{0t} \\ 0 & e^{0t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

Calculate $y(t)$ with e^{At}

$$y(t) = Ce^{A(t-t_0)}x(t_0) + C\int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \int_0^t \begin{bmatrix} 1 & t-\tau \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1/m \end{bmatrix} \tau^2 d\tau = \begin{bmatrix} 1 & 0 \end{bmatrix} \int_0^t \begin{bmatrix} (t-\tau)/m \\ 1/m \end{bmatrix} \tau^2 d\tau$$

$$= \int_0^t \frac{t-\tau}{m}\tau^2 d\tau = \int_0^t \frac{t\tau^2}{m} - \frac{\tau^3}{m} d\tau = (\frac{t}{3m}\tau^3 - \frac{\tau^4}{4m})|_0^t = t^4(\frac{1}{3m} - \frac{1}{4m}) = \frac{t^4}{12m}$$

Table of Contents

Discrete-Time Linear Time Invariant Systems

$$x(1) = Ax(0) + Bu(0)$$
$$x(2) = Ax(1) + Bu(1) = A^2x(0) + ABu(0) + Bu(1)$$
$$\vdots$$
$$x(k) = A^k x(0) + \sum_{m=0}^{k-1} A^{k-m-1} Bu(m)$$
$$y(k) = CA^k x(0) + \sum_{m=0}^{k-1} CA^{k-m-1} Bu(m) + Du(k)$$

Discretization of Continuous Time Systems

Forward Difference

$$\dot{x}(t) \approx \frac{x(t+T) - x(t)}{T}$$

This converts $\dot{x} = Ax + Bu$ into

$$\begin{aligned} x(t+T) &= x(t) + TAx(t) + TBu(t) \\ &= (I + AT)\,x(t) + TBu(t) \end{aligned}$$

Evaluating at $t = kT$

$x((k+1)T) = (I + TA)x(kT) + TBu(kT)$

$y(kT) = Cx(kT) + Du(kT)$

Zero Order Hold

The control output is kept as a constant within the sampling duration.

$u\left(t\right)=u\left(kT\right)=u\left(k\right)$ for $kT\leq t<\left(k+1\right)T$

$x(k)=e^{AkT}x\left(0\right)+\int_0^{kT}e^{A(kT-\tau)}Bu\left(\tau\right)d\tau$

$$\Rightarrow x(k+1)=e^{A(k+1)T}x\left(0\right)+\int_0^{kT+T}e^{A((k+1)T-\tau)}Bu\left(\tau\right)d\tau$$

$$=e^{AT}\left(e^{AkT}x\left(0\right)+\int_0^{kT}e^{A(kT-\tau)}Bu\left(\tau\right)d\tau\right)$$

$$+\int_{kT}^{(k+1)T}e^{A((k+1)T-\tau)}Bu\left(\tau\right)d\tau$$

Because $u(k)$ is a constant between kT and $(k+1)T$, let $\alpha=(k+1)T-\tau$,$d\tau=-d\alpha$

$\Rightarrow x(k+1)=e^{AT}x(k)+[\int_0^T e^{A\alpha}d\alpha]Bu(k)=e^{AT}x(k)+[\int_0^T e^{A\tau}d\tau]Bu(k)$

Zero Order Hold

$$\begin{aligned}
\int_0^T e^{A\tau} d\tau &= \int_0^T I + A\tau + \frac{A^2}{2!}\tau^2 + \ldots d\tau \\
&= I\tau + \frac{A}{2}\tau^2 + A^2 \frac{\tau^3}{3 \cdot 2!} + \ldots |_0^T \\
&= TI + \frac{T^2}{2} A + \frac{T^3}{3!} A^2 + \ldots \\
&= A^{-1} \left[TA + \frac{T^2}{2!} A + \frac{T^3}{3!} A + \ldots \right] \\
&= A^{-1}(e^{AT} - I)
\end{aligned}$$

Zero Order Hold

$$
\begin{aligned}
x(k+1) &= A_d x(k) + B_d u(k) \\
y(k) &= C_d x(k) + D_d u(k)
\end{aligned}
$$

$$A_d = e^{AT}, B_d = \int_0^T e^{A\tau} d\tau B = A^{-1}\left(e^{AT} - I\right) B = A^{-1}\left(A_d - I\right) B$$

$$C_d = C, D_d = D$$

Python can directly handle this: Take the example of longitudinal driving: COLAB LINK

Examples

$$\dot{x} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, y = \begin{bmatrix} 2 & 1 \end{bmatrix} x + 3u$$

a) Discretize with $T = 0.1$ s

$$A_d = e^{AT}, A = M\hat{A}M^{-1}$$

$$M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \widehat{A} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}, E_{2,1}(-1)M = I \Rightarrow M^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow A_d = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^{0.1} & 0 \\ 0 & e^{0.3} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1.11 & 0.24 \\ 0 & 1 \cdot 35 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 3 & -2 \\ 0 & 1 \end{bmatrix}, B_d =$$

$$A^{-1}(A_d - I)B = \frac{1}{3} \begin{bmatrix} 3 & -2 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 1.11 & 0.24 \\ 0 & 1 \cdot 35 \end{bmatrix} - I \right) \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.01 \\ 0.12 \end{bmatrix}, C_d = \begin{bmatrix} 2 & 1 \end{bmatrix}, D_d = 3$$

Examples

$$x\left(k+1\right)=\begin{bmatrix}1.11 & 0.24\\ 0 & 1.35\end{bmatrix}x\left(k\right)+\begin{bmatrix}0.01\\ 0.12\end{bmatrix}u\left(k\right)$$

$$y\left(k\right)=\begin{bmatrix}2 & 1\end{bmatrix}x\left(k\right)+3u\left(k\right)$$

Examples

b) solve $x(k)$ for $x(0) = [0 \quad 0]^T, u(k) = 3$

$$x\left(k\right) = A_d^k x\left(0\right) + \sum_{m=0}^{k-1} A_d^{k-m-1} B_d u(m),$$

$$A_d = M\widehat{A}M^{-1}, M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \widehat{A} = \begin{bmatrix} 1.11 & 0 \\ 0 & 1.35 \end{bmatrix}, M^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow A_d^k = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1.11^k & 0 \\ 0 & 1.35^k \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1.11^k & 1 \cdot 35^k - 1.11^k \\ 0 & 1.35^k \end{bmatrix}$$

Examples

$$\Rightarrow x(k) = \begin{bmatrix} 1.11^k & 1.35^k - 1.11^k \\ 0 & 1.35^k \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \underbrace{\sum_{m=0}^{k-1} \begin{bmatrix} 1.11^{k-m-1} & 1.35^{k-m-1} - 1.11^{k-m-1} \\ 0 & 1.35^{k-m-1} \end{bmatrix} \begin{bmatrix} 0.01 \\ 0.12 \end{bmatrix} 3}_{3\sum_{m=0}^{k-1} \begin{bmatrix} 0.12 \times 1.35^{k-m-1} - 0.11 \times 1.11^{k-m-1} \\ 0.12 \times 1.35^{k-m-1} \end{bmatrix}}$$

$$\Rightarrow y(k) = \begin{bmatrix} 6 & 3 \end{bmatrix} \sum_{m=0}^{k-1} \begin{bmatrix} 0.12 \times 1.35^{k-m-1} - 0.11 \times 1.11^{k-m-1} \\ 0.12 \times 1.35^{k-m-1} \end{bmatrix} + 9$$

Recap: Solutions to Linear Time Invariant State Equations

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Du\end{aligned} \Rightarrow \begin{cases} x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau \\ y(t) = Ce^{A(t-t_0)}x(t_0) + C\int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau + Du(t) \end{cases}$$

CT-LTI

Dicretization: $\Downarrow A_d = e^{AT}, B_d = \int_0^T e^{A\tau}d\tau B = A^{-1}\left(A_d - I\right)B$

$$\begin{aligned}x(k+1) &= A_d x(k) + B_d u(k) \\ y(k) &= Cx(k) + Du(k)\end{aligned} \Rightarrow \begin{cases} x\left(k\right) = A_d^k x\left(0\right) + \sum_{m=0}^{k-1} A_d^{k-m-1} B_d u\left(m\right) \\ y\left(k\right) = CA_d^k x\left(0\right) + \sum_{m=0}^{k-1} CA_d^{k-m-1} B_d u\left(m\right) + Du\left(k\right) \end{cases}$$

DT-LTI